Dynamical Galois Representations

by

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics at Brown University.

May 28, 2023

This dissertation by Ophelia M. S. Adams is accepted in its present form by the Department of Mathematics as satisfying the dissertation requirements for the degree of Doctor of Philosophy.

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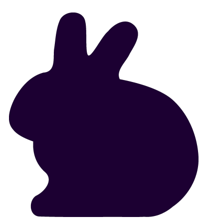
To my children, Cujo and Lula, and all the rabbits of the world.



Acknowledgments

First, I would like to thank my advisor, Joseph H. Silverman, for his guidance and encouragement throughout my time here at Brown, for our many conversations, and especially for introducing me to arithmetic dynamics. I would also like to thank my undergraduate advisor, Naomi Jochnowitz, for encouraging me to pursue mathematics as an undergraduate, and being a friend in the many years since.

Thank you to my family and friends, both in and out of mathematics, for keeping me afloat.

Lastly, I am grateful beyond words to my rabbits, Cujo and Lula, who chewed through my both life and so much mathematics: scratch paper, articles, books, and even computers. Their unwavering focus and clarity of vision inspire my mind, while their boundless joy and playful sense of mischief inspire my heart.

Preface

This dissertation discusses the ramification-theoretic behavior of Galois representations attached to dynamical systems over local fields, with applications to global fields. These arboreal representations are the dynamical counterparts to the Tate modules of abelian varieties, and one of the primary motivations for this work is developing dynamical analogues to what is already known about ramification for representations on the Tate module. In particular, we develop dynamical analogues of Sen's Theorem on the ramification filtration of Galois groups which are p-adic Lie groups and the Néron-Ogg-Shafarevich criterion for the good reduction of an abelian variety, as well as explore the extent to which the analogies fail. Additionally, we introduce "anabelian" dynamical representations on étale fundamental groups associated to dynamical systems

To be more precise, fix a field K, rational function $f(x) \in K(x)$ and base point $a \in K$. To construct the arboreal representation, form a graph of preimages of a by $f(x)$ with edges according to the action of $f(x)$. This graph is typically a tree, hence the arboreal appellation. This essentially coincides with the construction of the Tate module as systems of compatible inverse images of the identity by the multiplication-by-ℓ map, but we lose the algebraic structure of the object on which the absolute Galois group acts. Our anabelian representations come from a different incarnation of the Tate module: the (big) Tate module is isomorphic as a Galois module to the étale fundamental group of the abelian variety, and the Tate-module can be obtained by completing this group with respect to the group endomorphism induced by $[\ell]$. In a very similar way, we construct a geometric object X (an infinitely punctured projective line) from f such that $f : X \to X$ is an étale endomorphism. Then the usual étale fundamental group functor induces a functor from dynamical systems with a marked fixed point to profinite groups with a distinguished endomorphism with a Galois action. In a sense, these representations are more intrinsic to the dynamical system – the choice of a marked fixed point (i.e. base point of the étale fundamental group) only changes the group and representation by an inner automorphism, while the arboreal representations associated to a single rational function vary enormously as the base point is moved, and there is not yet a clear relationship between arboreal representations for the same function with different base points. In fact, one surprising outcome of the results of this dissertation is that some the aspects of the overall ramification behavior are independent of the base point, even if the representations are very different.

The study of these "anarboreal" representations is at an earlier stage, and they are somewhat more difficult to understand than arboreal representations. As we will see, these representations more closely reflect the geometry of the dynamical system, and also enable the use of powerful machinery from algebraic and anabelian geometry. One of the secondary goals of this dissertation is to serve as an introduction to these representations for dynamicists, and to provide a convincing test case for the ideas by applying them to our dynamical version of the Néron-Ogg-Shafarevich criterion and relating the structure of the anabelian representation to standard arboreal representations.

This thesis is based on published and submitted work of the author:

1. Mark O.-S. Sing (Mark A. Sweeney). A Dynamical Analogue of Sen's Theorem. *Int. Math. Res. Not. IMRN*, rnac070, March 2022.

2. Mark O.-S. Sing (Mark A. Sweeney). A Dynamical Analogue of the Criterion of Néron-Ogg-Shafarevich. Preprint: `https://arxiv.org/abs/2208.00359`. July 2022, updated November 2022.

Contents

Chapter 1

Introduction

In this dissertation, we state and prove dynamical analogues for two well-known results in arithmetic geometry: the Nerón-Ogg-Shafarevich (NOS) criterion for good reduction (Chapter 4) and Sen's Theorem on higher ramification (Chapter 5). Both of these connect the ramification-theoretic behavior of Galois representations to properties of the original object: the NOS criterion establishes a link between good reduction and unramified representations, while Sen's theorem shows that the higher ramification filtration and Lie filtration associated to a Galois representation are closely related. We prove our dynamical NOS criterion by classifying, under mild restrictions the (infinitely) ramified branch representations associated to a rational map, and we prove our dynamical analogue of Sen's theorem by explicitly calculating the Hasse-Herbrand function.

In addition, we introduce a new kind of dynamical Galois representation (Chapter 3). This construction is also motivated by the Tate module, but viewed through an "anabelian" lens, and very different in appearance from traditional arboreal representations, and in fact refines them. A much greater amount of input from algebraic geometry (the étale fundamental group) is necessary to define these representations, and it is the author's hope to convince the reader that these "anarboreal" representations are worth this extra technical cost. To this end, we will discuss how anarboreal representations seem to better detect geometric and arithmetic features of the dynamical systems from which they originate, and apply them to our formulation of the dynamical Néron-Ogg-Shafarevich criterion in Chapter 4.

It is worth noting that a preliminary version of the material in Chapter 4 purely in terms of arboreal representations was obtained without these ideas; the author then noticed the remarkable similarity between this result and Tamagawa's anabelian version of the Néron-Ogg-Shafarevich criterion, a connection which then developed into our anarboreal representations. The introduction of these ideas from anabelian geometry to the author's dynamical considerations led to a slightly stronger result and seems to better reveal the geometric features of the argument.

1.1 Background

Arithmetic dynamics is the study of dynamical systems from a number-theoretic perspective. The typical object in this study is a dynamical system $f : X \to X$, where X is a scheme and f a morphism defined over a field K or ring R of number-theoretic interests, such as $\mathbb{Q}$ or $\mathbb{Z}$ or even $\mathbb{Q}_p$. Typical questions might be about the distribution of prime divisors in orbits for f, or about p-adic analytic behavior of the dynamical system. In the present work, we are interested in certain Galois representations which can be attached to dynamical systems. Traditionally in arithmetic dynamics, the Galois representations which one studies are the arboreal representations (Definition 2.6) associated to a dynamical system with a marked point. These are typically viewed as dynamical analogues of the Galois representations associated to the Tate modules of abelian varieties.

Many guiding questions in arithmetic dynamics arise from or are inspired by analogies to well-studied objects in arithmetic geometry. Our aim is to develop dynamical analogues of the Néron-Ogg-Shafarevich (NOS) criterion and Sen's theorem.

The NOS criterion translates a geometric notion, good reduction, into more algebraic Galois-theoretic property: the criterion tells us that an abelian variety over a p-adic local field has good reduction at p if and only if the Galois action on the ℓ-adic Tate module is unramified for some (all) $\ell \neq p$. The criterion was first proven for elliptic curves independently by Ogg and Shafarevich, then later in full generality by Serre and Tate [33] using the powerful machinery of Néron models. Later, A. Tamagawa proved an anabelian analogue of the NOS criterion for affine curves [40] by replacing the Tate module with the étale fundamental group.

Sen noticed that, for Galois extensions whose Galois groups are p-adic Lie groups, there is a remarkable connection between the p-adic Lie filtration, which depends only on the Lie group, and the filtration by upper ramification subgroups: the two mutually refine each other in a precise way after a linear change of index [36]. This is an important input to p-adic Hodge theory, as it allows one to show that certain infinite extensions of fields are arithmetically profinite. Such extensions have a corresponding field of norms, from which the semilinear endomorphism Φ of the (Φ, Γ)-module associated to a p-adic representation is extracted.

1.2 Anarboreal Representations

In the analogy between dynamics and abelian varieties, arboreal representations are viewed as dynamical Tate modules. A source of the difficulty in studying arboreal representations is that they lack a counterpart for the Tate module's algebraic structure. With this in mind, Tamagawa's result is quite striking from the perspective of a dynamicist interested in arboreal representations: just like a dynamical system, a typical curve does not have any sort of algebraic structure, and so Tamagawa uses the pro-ℓ étale fundamental group as a replacement. This is exactly what we will endeavor to do in the the first part of this paper. Since we are concerned with dynamical systems on the curve $\mathbb{P}^1$, we will compare our dynamical systems to elliptic curves specifically.

Arboreal representations arise from the construction of the Tate module with the ℓ-torsion, or equivalently taking preimages of the identity $\mathcal{O}$ by $[\ell]$. But there are other constructions: the ℓ-adic Tate module is isomorphic to the ℓ-adic completion of the étale fundamental group of E. So an alternative way to create a dynamical analogue of the Tate module of a dynamical system $f : X \to X$ would be to find a related dynamical system whose endomorphism is *étale* and apply the étale fundamental group functor. The most straightforward way to do so is to delete all the points of X which prevent f from being étale: namely its grand critical orbit, the set of all points whose orbit intersects the critical orbit. With this in mind, our guiding analogies are as follows:

Elliptic Curves	Dynamical Systems on $\mathbb{P}^1$
	$\mathfrak{C}$, the grand critical pro-divisor
$U = E$	$U = \mathbb{P}^1 - \mathfrak{C}$
$[\ell] : U \to U$	$f : U \to U$
$[\ell]^* : \pi_1^{ét}(U_{\bar{K}}) \to \pi_1^{ét}(U_{\bar{K}})$	$\phi = f^* : \pi_1^{ét}(U_{\bar{K}}) \to \pi_1^{ét}(U_{\bar{K}})$
$T = \pi_1^{ét}(U_{\bar{K}})$	$\pi_1^{ét}(U_{\bar{K}})$

In other words, we view $f(x)$ and the multiplication-by-ℓ maps as étale endomorphisms of some variety, and use the geometric étale fundamental groups equipped with the induced endomorphisms as the big Tate module. For example, when $f(x)$ is a Lattés map associated to a quotient $\pi : E \to \mathbb{P}^1$ from an elliptic curve E and multiplication-by-ℓ map, the grand critical divisor of $f(x)$ is the image of $E[\ell^\infty]$ by π. It is not entirely clear what might be a dynamical counterpart to the small Tate modules T_ℓ. Certainly it suffices to consider the completion of $\pi_1^{ét}(U_{\bar{K}})$ at all primes smaller than

d, but a more dynamically natural alternative might be the quotient of $\pi_1^{ét}(U_{\bar{K}})$ by the subgroup

$$\Delta = \bigcap_{n \geq 1} N_G(\mathrm{img}(\phi^n)),$$

the intersection of the normal closures of the image of every iterate of ϕ. The endomorphism ϕ descends to the quotient, and for abelian varieties this construction recovers the ℓ-adic Tate module and its multiplication-by-ℓ map.

The reader should be careful to note that the arboreal representations with arbitrary base points *do not* correspond to the Tate module in our framework: it is only the grand critical representation, an intrinsic invariant of the dynamical system, which corresponds to the Tate module. This is not entirely surprising - for abelian varieties, the base point of the Tate module is always the identity. However, more general arboreal representations can still be incorporated into our anabelian framework. For example, it turns out that the grand critical arboreal representation exhibits (almost) all of the possibilities for ramification in any arboreal representation – which makes the grand critical representation a sort of ramification-theoretic upper bound.

1.3 Dynamical Néron-Ogg-Shafarevich

To help motivate the dynamical Néron-Ogg-Shafarevich theorem, let us recall two equivalences which play an important role in a proof of the Néron-Ogg-Shafarevich criterion for elliptic curves and isolate its dynamical content.

$$E \text{ has good reduction at } p$$

$$\Longleftrightarrow \tag{$*$}$$

$$\text{the reduction of } E[\ell^n] \text{ modulo } p \text{ has } \ell^{2n} \text{ distinct elements}$$

$$\Longleftrightarrow \tag{$**$}$$

the Galois action on the ℓ-adic Tate module is unramified

The dynamics we consider – rational functions on $\mathbb{P}^1$ – does not take place on a highly structured geometric object like the elliptic curve, and it is not necessarily clear what the dynamical analogue of an elliptic curve should be. As such, it is not clear that the equivalence $(*)$ has a purely dynamical interpretation in the arboreal framework. However, the equivalence $(**)$ amounts to a separability requirement for $[\ell^n]$: the preimage of O should have as many elements mod p as the degree of ℓ^n. This separabilty requirement naturally extends to the dynamical setting: when does the preimage of $\alpha \in \bar{K}$ by f^n have $\deg f^n$ distinct preimages in the residue field? When this happens, it follows immediately from Hensel's lemma that the associated arboreal representation is unramified, and our interest is in the converse.

At the same time, this kind of separability condition plays a prominent role in Tamagawa's anabelian analogue of the Néron-Ogg-Shafarevich theorem. In Tamagawa's theory, good reduction of a curve X punctured at a divisor D requires an extension of the pair (X, D) over K to a pair $(\mathfrak{X}, \mathfrak{D})$ over $\mathcal{O}_K$ where $\mathfrak{X}$ is smooth and $\mathfrak{D}$ is relatively étale, which recovers the original pair (X, D) on the generic fiber. One can immediately lift $\mathbb{P}^1_K$ to a smooth model $\mathbb{P}^1_{\mathcal{O}_K}$, and it is not difficult to check that the divisor D extends to a relatively étale divsior $\mathfrak{D}$ if and only if it has no components with multiplicity, and no points in its support reduce to the same point in the residue field after making a change of coordinate so that $0, 1, \infty$ lie in its support. Tamagawa shows that an affine punctured curve has good reduction in this sense if and only if the Galois action on the geometric étale fundamental group is unramified.

These observations suggest that the étale fundamental group may provide a bridge between

Tate modules and arboreal representations, and were the initial motivation for the anabelian constructions discussed earlier. In this case, an infinitely punctured projective line takes the place of the elliptic curve. This suggests a strong notion of good reduction: points of the pre-critical locus over K should not collapse when reduced modulo p. In other words, we require that a dynamically interesting invariant not degenerate modulo p, which is very much in the spirit of good reduction.

The logical structure of the various components of the dynamical version of the Néron-Ogg-Shafarevich is as follows. The Galois group of the ground field K is denoted by Γ, the prime p is the residue characteristic, and $\ell \neq p$ is an auxiliary prime.

The Γ action on $\bar{\Pi}^{(\ell)}$ is ramified.

$\Updownarrow$ Theorem 4.1

The pre-critical incidence portrait has a cycle.
This cycle is either
directed or undirected

directed	undirected
Theorem 4.8 $\Updownarrow$	$\Updownarrow$ Theorem 4.9
There is an infinitely ramified branch near a pre-critical branch.	There is a finitely ramified branch near a pre-critical branch.
Theorem 4.8 $\Updownarrow$	$\Updownarrow$ Theorem 4.9
Some branch is infinitely ramified.	Some branch is finitely ramified.

In Theorem 4.8 the infinitely ramified branch can be taken to be arbitrarily close to the pre-critical branch. If no critical points are periodic then one can take the infinitely ramified branch to be pre-critical. In contrast, for Theorem 4.9 the finitely ramified branch *cannot* necessarily be chosen to be arbitrarily close to the pre-critical branch, and in general lies on a p-adic annulus around the pre-critical branch.

1.4 Dynamical Sen

In the dynamical setting, we replace p-adic Lie groups and the Lie filtration of Sen's theorem with "branch extensions" and their "branch filtration". For those familiar with arboreal representations, we are taking a single branch of the tree, filtered by height up the branch. Thus our dynamical version of Sen's theorem says that, after possibly extending the ground field and making a linear change in index, each member of the branch filtration coincides exactly with a member of the upper ramification filtration.

The upper ramification filtration is in general quite difficult to understand, and captures subtle arithmetic phenomena, while the branch filtration is quite simple and dynamically natural: starting from our ground field K, we have a tower of extensions K_n over K obtained by adjoining a compatible sequence ("branch") of preimages of the base point. We are able to give a general sufficient criterion for our result to hold: it applies to extensions associated to so-called "strictly tamely ramification-stable" branches. In our situation, "tamely" simply means that p does not divide a certain quantity d, which is the limiting valuation of the members of the branch. Such branches are particularly striking from a dynamical perspective, exhibiting a kind of stability in the structure of their higher ramification: the intermediate Hasse-Herbrand functions associated to K_n/K_{n-1} are identical up to small and well-controlled errors, neglecting scaling. For these branches, we obtain our main result:

Theorem 5.18. *Suppose our branch, associated to the polynomial $f(x)$ and base point α_0, is strictly tamely ramification-stable over K. Then K_∞/K is arithmetically profinite, and there is a constant V such that for all n,*

$$K_n = K_\infty^{((V-1)n+1)}.$$

A more literal, and weaker, analogue of Sen's theorem in the dynamical setting would be that the two filtrations refine each other, again, after a linear change of index. However, for one of our applications, to a question of Berger [8], we need the stronger formulation of Theorem 5.18.

We are able to give a general sufficient criterion for a branch to be strictly tamely ramification-stable, Proposition 5.11. This criterion consists of two pieces: that p does not divide d, and verifying an inequality depending only on the valuations of the coefficients of $P(x)$ and the valuation of α_0. Some branches which are not tamely ramification-stable may become so after extending the ground field and re-indexing the branch; we call such branches *potentially* tamely ramification-stable.

Using this criterion, we are able to show that if $f(x)$ is either post-critically bounded or of prime degree p, and we take a branch such that p does not divide the associated constant d (a limiting normalized valuation of the members of the branch), then it is *potentially* tamely ramification-stable, and we use this information to characterize higher ramification in the associated extension:

Corollary 5.19. *Let $P(x)$ be a polynomial which either has degree p, or is post-critically bounded and has degree p^r. Take any nontrivial branch for $P(x)$, and suppose p does not divide the constant d associated to the branch. Then the dynamical branch extension K_∞/K is arithmetically profinite, and there are constants N and V such that after replacing K by K_N,*

$$K_n = K_\infty^{((V-1)(n-N)+1)},$$

for all n.

For any particular branch, it is not difficult to apply our criteria to check whether or not it is (potentially) strictly tamely ramification-stable, so long as one knows that p does not divide d. In fact, our criterion is almost entirely effective: only the stipulation that p does not divide d is not known to be effective. Each branch determines certain "limiting ramification data" from which one can completely recover the Hasse-Herbrand function of the associated branch extension in the strictly tamely ramification-stable case when d is known. The calculation of the limiting ramification data depends only on $P(x)$ and some of the initial entries of the branch (the number of entries needed is itself effective). While we lack a general algorithm to determine d, it can be calculated in many particular instances.

We apply our results to provide a partial answer to two questions. One is raised by Berger [8], who asks: is it possible to show by elementary methods that if K_∞/K is Galois and the base point is a uniformizer then its Galois group is abelian? This is known to be true by Berger [9] using quite sophisticated methods from p-adic Hodge theory. Our main theorem involves more elementary tools, and allows us to re-prove this fact in some situations:

Theorem 5.20. *Assume p is odd. Suppose α_0 is a uniformizer for K, $f'(0)$ is nonzero, and we are given a branch associated to $f(x)$ and α_0 which is tamely ramification-stable.*

If K_∞/K is Galois, it is also abelian.

The other question is suggested by both Aitken, Hajir, and Maire (Question 7.1 in [1]) and Bridy, Ingram, Jones, Juul, Levy, Manes, Rubinstein-Salzado, and Silverman (Conjecture 6 in [10]), who essentially ask if it is possible for an arboreal extension over a number field to be ramified at finitely many primes *but not wildly ramified.* It turns out that this is not possible for polynomials of prime-power degree:

Theorem 5.22. *Let F be a number field and $\mathfrak{p}$ a prime of F lying over a rational prime p. Let $P(x) \in \mathcal{O}_F[x]$ be a monic polynomial of degree p^r such that $f(x) \equiv x^{p^r} \mod \mathfrak{p}$, and let $\alpha_0 \in F$.*

Then the arboreal representation associated to $f(x)$ and α_0 is infinitely wildly ramified.

If, further, $f(x)$ has prime degree and $v(\alpha_0) \neq 0$, or is post-critically bounded with no restriction *on $v_{\mathfrak{p}}(\alpha_0)$, and there is a branch over α_0 whose associated constant d is not divisible by p, then every higher ramification subgroup over $\mathfrak{p}$ of the arboreal representation is nontrivial.*

1.5 Prior Work

Ramification has always played an important role in the study of arboreal representations, beginning with the introduction of arboreal representations by Odoni [27]. For example, progress towards various versions of Odoni's Conjecture [7, 21, 38] make essential use of ramification-theoretic methods. More recently, ramification in arboreal extensions has been studied in its own right [3, 36, 18, 35, 1, 8, 39], with interesting applications to dynamical systems over both global and local fields. To the author's knowledge, there has been no prior work on the Néron-Ogg-Shafarevich criterion in the dynamical setting, nor work on interactions between anabelian geometry and dynamics.

While the main results of this dissertation are described as dynamical analogues of the NOS criterion and Sen's theorem, our initial motivation actually comes directly from arithmetic dynamics and the structure of arboreal representations associated to post-critically finite maps. Arboreal representations, first introduced by Odoni [27], have been a subject of significant focus in arithmetic dynamics. Odoni exhibited two specific examples of polynomials with maximal arboreal representations: a monic polynomial with generic coefficients over any field, and the polynomial $x^2 - x + 1$ over $\mathbb{Q}$. Odoni's former result led him to state a conjecture–now known to be false by work of Dittman and Kadets [13]–that examples of such polynomials exist over any Hilbertian field. Since then, others have stated and studied a multitude of variations on Odoni's conjecture. This recently culminated in the resolution of (one version of) Odoni's conjecture over number fields, in prime degree by Looper [21], in all even degrees and certain odd degrees by Benedetto and Juul [7], and finally for all degrees by Specter [38]. The branch extensions we tackle are the subextensions of the full arboreal extension which are associated to a single branch of the full preimage tree. The extensions we study appear within the full arboreal representation and the ramification along such branches is quite important for the aforementioned results on Odoni's conjecture. Additionally, Andrews and Petsche [4] as well as Ferraguti and Pagano [16] have also used ramification information to prove interesting results about *abelian* arboreal representations over number fields. Our results are finer than necessary for any of the papers mentioned, but the important role ramification plays in those results suggests the potential value of the more detailed and delicate ramification information that we obtain. Though arboreal extensions over global fields are still quite mysterious, even less is known over local fields. Anderson, Hamblen, Poonen, and Walton [3] studied full arboreal extensions in the local setting for polynomials of the form $x^n + c$. In fact, they produce an example which shows that a literal dynamical analogue of Sen's theorem cannot hold in full generality, even in the case of prime degree. Very recently, Hamblen and Jones [18] also studied higher ramification in arboreal representations with Sen's theorem in mind and proved that various interesting families of arboreal representations are deeply wildly ramified by showing that they contain all the p-power roots of unity, as well as a different dynamical version of Sen's theorem. The results in this dissertation and those of Hamblen-Jones are, in a sense, orthogonal generalizations of Sen's theorem: Hamblen and Jones restrict the base point but not the rational map, while the author's result restricts the rational map but not the base point. Jones and Hamblen were unaware of the author's earlier paper [36] or corresponding preprint, and the results are entirely independent.

The case of post-critically bounded polynomials is of particular dynamical interest because it includes the post-critically finite polynomials of prime-power degree. Currently, the arboreal representations of post-critically finite polynomials are not well-understood, but it is known that they have arboreal representations which are ramified at only finitely many primes [1], so one would expect their arboreal representations to largely be controlled by their local behavior at those primes. Our result reveals initially unexpected structure to their wild ramification at the prime in question.

Some other work has been done with extensions of the kind we consider. Both Berger [8] and Cais and Davis [11] study them (under the name "ϕ-iterate extensions") with the machinery of p-adic Hodge theory, and show that if these extensions are Galois they must be abelian. Cais, Davis, and Lubin [12] study the ramification in a somewhat more general setting, using similar methods to ours to give a characterization of arithmetically profinite extensions – it is an important corollary of Sen's theorem that p-adic Lie extensions are arithmetically profinite. The dynamical case of their result applies to a broader class of polynomials than ours, with the restriction that the base point is a uniformizer. For the polynomials considered in this paper, we are able to relax this restriction on the base point and obtain more precise information about the ramification of our extensions.

Chapter 2

Preliminaries

In this chapter we gather some of the preliminary facts, definitions, and notation which form a common basis for Chapters III, IV, and V. Mainly, we introduce branch representations and the ramification behavior of branch representations contained in a periodic open disc. The contents are the respective dynamical preliminary sections of [37, 36].

2.1 Basic Definitions and Notation

Let p be a prime and K a mixed-characteristic p-adic field with ring of integers $\mathcal{O}_K$ and a uniformizer π_K and residue field $\tilde{K}$. In other words, $\mathcal{O}_K$ is a complete discrete valuation ring with perfect residue field $\tilde{K} = \mathcal{O}_K/(\pi_K)$ of characteristic p, and K is the field of fractions of $\mathcal{O}_K$. For example, K might be a finite extension of $\mathbb{Q}_p$ or $\mathbb{Q}_p^{ur}$. In some situations we may require that the residue field $\mathcal{O}_K/(\pi_K)$ be finite.

Definition 2.1. Let S be a scheme. A **dynamical system** over S consists of an S-scheme X equipped with a morphism $f : X \to X$ over S.

The nth iterate of f is denoted by f^n. Typically, f will be a rational map on some projective space. In this case, we fix a choice of polynomials $p, q \in \mathcal{O}_K[x]$ such that $f(x) = p(x)/q(x)$ and at least one coefficient of p or q is a unit. This choice is unique up to multiplying q and p by the same unit constant in $\mathcal{O}_K$, so the (multi)set of valuations of the coefficients of the polynomials p and q

depends only on f. We refer to these as the valuations of the coefficients of f.

Let $\bar{K}$ be an algebraic closure of K and v an extension of the valuation on K to $\bar{K}$ normalized so that $v(p)$ and the valuations of the coefficients of f are integers. We fix an auxiliary subfield E of K such that $v(E) = \mathbb{Z}$. Typically one begins with $E = K$ and v the usual normalized valuation, but the field K will be allowed to vary while E is held fixed. This essentially amounts to a choice of valuation v on $\bar{E} = \bar{K}$, independent of K, such that p and the coefficients of the rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ have integer valuation.

Let Γ_K be the absolute Galois group of $\bar{K}$ over K, and denote by Γ_K^{ν} the higher ramification group associated to a nonnegative real number ν by the upper-numbering. For detailed exposition on higher ramification, see Serre [34] or Lubin [22]. Note that the upper and lower filtrations are indexed slightly differently between these two sources; we follow Lubin's convention.

When f is a power series, we denote by f_i the coefficient of x^i in $f(x)$.

In this dissertation, we are almost exclusively interested in dynamical systems on curves. Since curves of genus 2 or greater have few endomorphisms, all of which are finite-order automorphisms, the most interesting dynamics occurs on curves of genus 0 or genus 1; in the presence of a rational point, this leaves just projective lines or elliptic curves. The study of elliptic curves is a rich and well-developed field in its own right. Our attention is directed at dynamical systems on the projective line over $\operatorname{Spec} K$ or $\operatorname{Spec} \mathcal{O}_K$, and one of our main goals is to understand the extent to which this theory parallels that of the study of elliptic curves. As such, we will often assume $X = \mathbb{P}^1_K$ and f is a rational map, hence an endomorphism of $\mathbb{P}^1_K$.

2.2 Good Reduction I

An important notion in dynamical systems is good reduction:

Definition 2.2. Let $f : X \to X$ be a dynamical system over K. Then we say that the dynamical system has **good reduction** if there is a proper, smooth $\mathcal{O}_K$-scheme $\mathfrak{X}$ and a dynamical system $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$ over $\mathcal{O}_K$ whose restriction to the generic fiber is the original dynamical system.

When a dynamical system has good reduction, the restriction $\tilde{f} : \tilde{X} \to \tilde{X}$ to the special fiber $\tilde{X}$ is a dynamical system over the residue field $\tilde{K} = \mathcal{O}_K/(\pi_K)$.

In general, we adopt the convention of using Latin letters for objects defined over $\operatorname{Spec} K$ and gothic letters for those defined over $\operatorname{Spec} \mathcal{O}_K$, while a tilde over an object denotes reduction.

We are typically interested in $X = \mathbb{P}^1_K$ and $\mathfrak{X} = \mathbb{P}^1_{\mathcal{O}_K}$, which has reduction $\tilde{X} = \mathbb{P}^1_{\tilde{K}}$. To be more concrete, $\mathfrak{X}$ is obtained by gluing $\operatorname{Spec} \mathcal{O}_K[x]$ and $\operatorname{Spec} \mathcal{O}_K[y]$ along the open subschemes $\operatorname{Spec} \mathcal{O}_K[x, 1/x]$ and $\operatorname{Spec} \mathcal{O}_K[y, 1/y]$ by the isomorphism $x \mapsto \frac{1}{y}$. It is important to note that a point $a \in X$ has a canonical lift to a point $\mathfrak{a}$ in $\mathfrak{X}$ by the valuative criterion for properness. Concretely, upon fixing coordinates, if a is integral, let $\mathfrak{a} = a$ as an ideal in $\mathcal{O}_K[x]$, and if a is not integral, let $\mathfrak{a}$ be the ideal corresponding to $1/a$, taken in $\mathcal{O}_K[y]$. In this case, a non-integral point of X reduces to the point at ∞ of $\tilde{X} = \mathbb{P}^1_{\tilde{K}}$.

One quite often studies dynamical systems with a marked divisor – for instance, arboreal representations are associated to dynamical systems *with a marked base point*. The notion of good reduction can be extended to this case:

Definition 2.3. Suppose that $f : X \to X$ is a dynamical system over K with an effective (Cartier) divisor D on X defined over K. We say that the pair $(f : X \to X, D)$ has good reduction if

$f : X \to X$ has good reduction in the sense of Definition 2.3, hence a lift $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$, and there is a relatively étale divisor $\mathfrak{D}$ of $\mathfrak{X}$ which restricts to D on the generic fiber.

In our usual case, $X = \mathbb{P}^1_K$ and $\mathfrak{X} = \mathbb{P}^1_{\mathcal{O}_K}$, this notion can be made very concrete: a pair (f, D) of a dynamical system and divisor on X has good reduction if and only if f has good reduction, D is reduced, and no two points in the support of D have the same reduction in $\tilde{X}$ (after making a change of coordinate so that 0, 1, and ∞ are in the support of D). One should be cautious when comparing this definition to the literature, where it is typical to say that a dynamical system with a marked base point (f, a) has good reduction if and only if f has good reduction and a is integral (i.e. in $\mathcal{O}_K$). Our notion of good reduction does not require that the base point a be integral. This is consistent with the author's observation [36] while studying higher ramification that non-integral base points lead to essentially the same behavior with only small adjustments arising from the change in sign.

2.3 Branch Representations

Instead of the full arboreal representation, we tend to work with more manageable sub-representations.

Definition 2.4. A **branch** for f over K is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of elements of $\bar{X}$, such that $f(\alpha_{n+1}) = \alpha_n$. The first entry, α_0, is called the **base point**.

Given any branch, its Galois orbit naturally has the structure of a directed graph, which is typically a tree, though in general it can have a cycle. The natural coordinate-wise action of Γ_K on this tree gives rise to a kind of representation:

Definition 2.5. Given a branch $B = (\alpha_n)_{n\in\mathbb{N}}$ for f, let T be the graph formed by the Galois orbit of B. The associated **branch representation** is the homomorphism from Γ_K to $\mathrm{Aut}(T)$ induced by the action of Γ_K.

Definition 2.6. Given a base point $a \in K$, let T be the graph of all preimages of a by $f(x)$, with directed edges according to the action of f. Then the **(full) arboreal representation for** $f(x)$ **and** a is the homomorphism from Γ_K to $\mathrm{Aut}\, T$ given by the Galois action on the graph.

The full arboreal representation can naturally be decomposed in terms of the branch representations associated to the maximal Galois orbits of branches. Over global fields, branch representations tend to be quite large: a very weak form of Odoni's conjecture predicts that the Galois action on branches of a "typical" arboreal representation is transitive, and hence the branch representation coincides with the full arboreal representation. Over local fields this behavior is not typical.

Given a branch $B = (\alpha_n)$, there is a naturally associated tower of fields $K_n = K(\alpha_n)$ and $K_\infty = \bigcup_n K_n$. The extension K_∞/K is unramified (resp. finitely ramified, resp. infinitely ramified) if and only if the corresponding branch representation is unramified (resp. finitely ramified, resp. infinitely ramified). This also gives rise to a dynamical filtration on Γ_K, the absolute Galois group of K:

Definition 2.7. Fix a branch $B = (\alpha_n)$ and let $K_n = K(\alpha_n)$. Then we define the n**th branch subgroup of** Γ_K **with respect to** B as the stabilizer of K_n in Γ_K. This subgroup is denoted $\Gamma_{K,n,B}$ or simply Γ_n when B is understood.

These subgroups are not to be confused with the lower numbered ramification subgroups of finite Galois extensions of K. Since the absolute Galois group does not admit a lower numbering, no confusion should arise. The branch filtration does behave like the lower numbering filtration insofar as it is compatible with restriction to subgroups – if L is an extension of K, then $\Gamma_{L,n,B} =$

$\Gamma_{K,n,B} \cap \Gamma_L$.

2.4 Power Series

The branches of greatest interest to us are those contained entirely in an open p-adic disc around a fixed point of $f(x)$. We define discs with the spherical metric on $\mathbb{P}^1$ – it is the natural choice in our setting, as it is conjugation invariant. Away from ∞ the spherical metric coincides with the usual p-adic metric. On such a disc, the rational function can be replaced by a convergent power series with integral coefficients and finite Weierstrass degree. This form is much more convenient for the study of ramification.

Definition 2.8. We say that a **branch** $B = (\alpha_n)$ **is periodic** if its entries are periodic. Likewise if $\tilde{B} = (\tilde{\alpha}_n)$ is periodic we say that B is **residually periodic**. The **(exact) period** m of B is the smallest integer m such that $\alpha_n = \alpha_{n+m}$ for all n, and the **(exact) residual period** of B is the (exact) period of $\tilde{B}$.

Definition 2.9. Suppose a branch $B = (\alpha_n)_{n\in\mathbb{N}}$ is residually periodic of period m. Define the **ramification index of the branch**, denoted e_B, as the Weierstrass degree of $f^m(x)$ expanded as a power series around $x = \alpha_0$.

The ramification index of a residually fixed branch is constant on the open disk of (spherical) radius 1 around the branch, so it suffices to expand $f(x)$ around any point near α_0.

Example. Over $\mathbb{Q}_p$ the polynomial $x^p + px^2 - p^2x + p$ has a unique branch based at $\alpha_0 = p$ and contained in the open unit disk. This branch has residual period 1 (i.e. is residually fixed) and has ramification index p. The unique branch at the totally ramified fixed point ∞ also has ramification index p.

Example. Let n be a positive integer not divisible by p. The rational map

$$f(x) = \frac{x^n - px}{1 - x}$$

has a unique branch based at $\alpha_0 = p$ and contained in the open disc. This branch is also residually fixed and has ramification index n: the spherical distance between 0 and p is less than 1, and $f(x) = (x^n - px)(1 + x + x^2 + ...)$ when expanded as a power series around $x = 0$, which clearly has Weierstrass degree n.

This rational map has a fixed point at ∞ as well, but the ramification index of the unique branch contained in the open disc around ∞ (in the chordal metric) is $n = 1$. To see this, make the change of coordinate $x \mapsto 1/x$ to move ∞ to zero. In these coordinates, our rational map is

$$\frac{1}{f\left(\frac{1}{x}\right)} = \frac{x^n - x^{n-1}}{1 - px^{n-1}},$$

which evidently has Weierstrass degree $n - 1$ on the disc around $x = 0$.

The reduction of the the first example, $f(x) = x^p + px^2 - p^2x + p$, is the rather special map $\tilde{f}(x) = x^p$. The derivative of $\tilde{f}$ is identically zero, or in other words $\tilde{f}$ is ramified everywhere. This special behavior sometimes needs to be handled separately:

Definition 2.10. The **height** of $f(x)$ is the largest integer h such that we can write $\tilde{f}(x) = \tilde{Q}(x^{p^h})$ for some rational function $Q(x)$. A rational function has positive height if and only the reduction of its derivative is identically zero. For simplicity, we fix a choice of Q_f for f such that $\tilde{f}(x) = \tilde{Q}_f(x^{p^h})$, where h is the height of f.

The significance of height zero is the following fact, immediate from the definitions:

Proposition 2.11. *If $f(x)$ has height zero, then every critical point of its reduction $\tilde{f}(x)$ is the reduction of a critical point of $f(x)$.*

This is trivially false when the rational map has positive height, as then $\tilde{f}'(x) = 0$ and every point is a critical point.

Our use of the word "height" is in reference to the height of a formal group, in the spirit of the analogy between arithmetic dynamics and abelian varieties. The reductions of rational maps with positive height exhibit vague behavioral similarities with positive height endomorphisms of formal groups.

2.5 Main Ramification Lemma

The following proposition is the starting point for the main results in Chapters IV and V. In Chapter IV, one of our main accomplishments is a kind of converse, while in Chapter V we improve this relatively coarse description of ramification by explicitly describing the higher ramification filtration of the representation.

We only apply Lemma 2.12 to the power series expansions of rational maps, but the more general case fits naturally into the study of dynamical systems on the open p-adic unit disk initiated by Lubin [24, 23].

Lemma 2.12. *Let $f(x)$ be a power series in $\mathcal{O}_K[[x]]$ with finite Weierstrass degree e and such that $f(0) = 0$. Let $(\alpha_n)_{n\in\mathbb{N}}$ be any nontrivial branch for $f(x)$ contained in the open unit disk. Then for all n sufficiently large,*

(a) $v(\alpha_{n+1}) = \dfrac{v(\alpha_n)}{e}$,

(b) the sequence $(e^n v(\alpha_n))_{n\in\mathbb{N}}$ is eventually constant,

(c) $K(\alpha_{n+1})/K(\alpha_n)$ is totally ramified of degree e.

Proof. When $e = 1$ all of the claims are trivial – from the Newton polygon it is immediately apparent that $v(\alpha_n) = v(\alpha_{n-1})$, while Hensel's lemma guarantees $\alpha_n \in K(\alpha_{n-1})$. So assume $e > 1$.

Let π_n be a uniformizer for $K_n = K(\alpha_n)$ and write $\alpha_n = u_n \pi_n^{d_n}$ for some integer d_n (possibly negative) and unit u_n of $\mathcal{O}_{K_n}$.

(a) By considering the Newton polygon again, we see that

$$v(\alpha_{n+1}) \leq \max\left\{v(\alpha_n) - 1, \frac{v(\alpha_n)}{2}\right\},$$

and hence the sequence $v(\alpha_n)$ decreases monotonically to zero. Then there is $N > 0$ such that for all $n \geq N$, $v(\alpha_n)$ is strictly smaller than the valuation of each of the first $e - 1$ coefficients. Therefore the Newton polygon has a single line of negative slope $-v(\alpha_n)/e$, hence $v(\alpha_{n+1}) = \frac{v(\alpha_n)}{e}$.

(b) Immediate from (a).

(c) Let e_n be the ramification index of K_n/K_{n-1}. Take $n - 1$ large enough that (a) holds, and so we have both

$$v(\alpha_n) = v(u_n \pi_n^{d_n}) = d_n v(\pi_n) = \frac{d_n v(\pi_{n-1})}{e_n},$$

and

$$v(\alpha_n) = \frac{v(\alpha_{n-1})}{e} = \frac{v(u_{n-1}\pi_{n-1}^{d_{n-1}})}{e} = \frac{d_{n-1} v(\pi_{n-1})}{e}.$$

Comparing the two yields the following relation:

$$d_n = \frac{e_n}{e} d_{n-1}. \tag{2.1}$$

From (2.1), we see that if $e_n = e$, then $d_n = d_{n-1}$. So we need only verify that $e_n = e$ for n large enough. Evidently $e_n \leq e$, so we wish to show that this inequality is strict at most finitely often. Indeed, each time the inequality is strict, the integer d_n has strictly fewer divisors than d_{n-1}. An integer cannot have a negative number of divisors, so these strict drops happen only finitely many times. □

Lemma 2.13. *Let $f(x)$ be a rational map over K and B a branch for f. If B is residually periodic of period m and ramification index e, then for all n sufficiently large, the extension K_{m+n}/K_n is totally ramified of degree e.*

Proof. Replace f by f^m and B by $(\alpha_{km+i})_{k\in\mathbb{N}}$, where $i < m$ and $i = n \bmod m$ so that B is residually fixed and contains α_n. Change coordinate so that B is contained in the open unit disc and f fixes 0. Expand f as a power series then apply Lemma 2.12. □

A related result, under the assumption $p \nmid e$, is proven by Ingram [19], who exploits the fact that polynomials have a totally ramified fixed point at ∞. After making a change of coordinate to move ∞ to 0, the resulting rational function can be written as a power series with Weierstrass degree equal to its degree, to which Lemma 2.12 applies. When $p \neq e$, Ingram shows that f is analytically conjugate to a power map. His construction can be directly adapted to power series whose Weierstrass degree is not divisible by p. Of course, for our dynamical Sen's theorem, we require $p|e$. In some circumstances, it is possible to show that f is analytically conjugate to a powering map or series with integral coefficients: Salerno and Silverman [32] study this when e is a power of p.

Chapter 3

Étale Fundamental Groups

In this chapter, we describe conditions under which a dynamical system $f : X \to X$ over K on a general scheme X can be modified to a dynamical system whose endomorphism is finite étale, and hence induces a dynamical system on a profinite group by taking the étale fundamental group. There is a Galois action on this profinite group which is equivariant with respect to the dynamics, and can be viewed as a dynamical analogue of the Galois action the (big) Tate module equipped with its multiplication-by-ℓ endomorphism. This anabelian representation refines the usual arboreal representation, and we will explain how to reconstruct the latter from the former.

This material is drawn from the author's preprint [37], primarily Section 2 and some pieces of Section 4. It was subsequently brought to the author's attention that a related construction has been studied by Pink [30, 29, 28]. Our fundamental groups are much larger than those considered by Pink because we want f to not merely be an étale cover of another fixed scheme, but to further be an *endomorphism* of this scheme so that we may functorially assign a dynamical system on a scheme to a dynamical system on a profinite group.

3.1 Dynamical Purity

Dynamical considerations typically give rise to infinite diagrams of schemes, and then to infinite projective limits in the category of schemes. Such limits need not exist in general, but they will when the morphisms in the diagram are affine. So we restrict our attention to dynamical systems in which we can take such limits

Definition 3.1. Fix a dynamical system $f : X \to X$ with f finite and flat. Let B_n be the branch locus of f^n, the image of the ramification locus, and D_n the union of B_n with its inverse image by f^n, then set $U_n = X - D_n$. Then we say that the dynamical system is **nearly étale** if the inclusion of U_n into X is an affine morphism.

We can see that $U_{n+1} \subseteq U_n$, hence the inclusion of U_{n+1} into U_n is also affine.

It is worth noting that for sufficiently nice schemes X with a finite endomorphism $f : X \to X$, the associated dynamical system will typically be nearly étale. This is true of all curves, and in fact any curve over K is itself affine after making at least one puncture at a K-rational or closed point. Indeed, it can be shown that in many cases the purity of the ramification locus implies that inclusion of the complement of the ramification locus of $f^n(x)$ is affine [5]. We delete somewhat more, the entire inverse image of the branch locus B_n, to ensure that $f : U_{n+1} \to U_n$ is étale. In this sense, being nearly étale is a strong dynamical purity condition.

Proposition 3.2. *If $f : X \to X$ is nearly étale, then the restriction $f : U_{n+1} \to U_n$ is finite étale.*

Proof. Consider an open affine $\operatorname{Spec} A$ in U_n. Because f and the inclusions of U_n and U_{n+1} in X are affine, the inverse image of $\operatorname{Spec} A$ by f is an affine open $\operatorname{Spec} B$ in X. Moreover, $f^{-1}(U_n) = U_{n+1}$ and so $\operatorname{Spec} B$ is an open affine in U_{n+1}. Since f is finite and flat and the ramification locus has been deleted, the extension of rings B/A induced by f is finite étale, so we are done. □

Curves of genus at least two have very few endomorphisms, so dynamics on curves is essentially limited to genus zero and genus one: projective lines and elliptic curves. Elliptic curves have their own well-developed theory, so we are primarily concerned with the situation in genus zero (though often with elliptic curves in mind for motivation). This leaves us only with projective lines, and here every finite endomorphism is nearly étale:

Proposition 3.3. *Let A be an integral domain and $f : \mathbb{P}^1_A \to \mathbb{P}^1_A$ a dynamical system defined by a nonconstant rational map f over A. This dynamical system is a nearly étale dynamical system.*

Proof. Morphisms associated to nonconstant rational maps are finite, and upon dehomogenizing it is easy to see that the inclusion of each $U_n = \mathbb{P}^1_A - D_n$ into $\mathbb{P}^1_A$ is affine (in fact, each U_n is already an open affine subscheme of $\mathbb{P}^1_A$). □

3.2 Pro-Divisors

Just as dynamical considerations require us to consider infinite diagrams of schemes, we will also want to consider divisors (arising, for instance, from orbits or backward orbits) which have infinitely many points. This requires a small generalization from divisors to "pro-divisors", to allow infinite formal sums with possibly infinite coefficients:

Definition 3.4. Let X be a scheme. An **effective pro-divisor** D on X over K is a sequence of effective Cartier divisors $(D_n)_{n \in \mathbb{N}}$, defined over K, such that for all $m \leq n$, the scheme-theoretic intersection of D_n and D_m is D_m. Equivalently, for $m \leq n$, we have $\operatorname{Supp} D_m \subseteq \operatorname{Supp} D_n$ and the multiplicity of every point P in D_n is greater than or equal to its multiplicity in D_m.

A general philosophy in dynamics is that the critical orbit controls the behavior of the dynamical system; we are also working "backwards" and so we work with the grand critical orbit:

Definition 3.5. The **grand critical pro-divisor of** f is the effective pro-divisor GC_f on X determined by the system of divisors $D_n = B_n + f^{-n}(B_n)$ where B_n is the branch locus of $f^n(x)$ and $f^{-n}(B_n)$ is the preimage of B by f^n.

The complement of C_f "should be" the largest open subscheme on which all iterates of f^n are étale and surjective. Of course, this complement is neither open nor a scheme in general. However, the complement of the grand critical divisor of a nearly étale endomorphism is a scheme, though generally not an open subscheme of the original.

Theorem 3.6. *Let* $f : X \to X$ *be a nearly étale dynamical system, with ramification divisors* R_n *and open subschemes* $U_n = X - D_n$*. Then the associated limit* $U_\infty = \varprojlim U_n$ *of the complements of the branch and ramification divisors with their natural inclusions exists in the category of schemes, and* $f : U_\infty \to U_\infty$ *is finite étale.*

Proof. The inclusion of each U_n into X is affine, hence also its inclusion into U_{n-1}. Thus the morphisms in the diagram are all affine, and in this case such a limit does exist in the category of schemes. The restrictions $f : U_n \to U_{n-1}$ are all finite étale, and so they have a finite étale limit $f : U_\infty \to U_\infty$ as well. □

3.3 Critical Incidence Graphs

The grand critical pro-divisor is an important invariant of a dynamical system $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$. A closely related invariant is the full critical locus of the dynamical system

Definition 3.7. The **full critical locus** of a dynamical system $f : X \to X$ is the pro-divisor determined by the system $(R_n)_{n\in\mathbb{N}}$ of ramification divisors: R_n is the critical locus of f^n, with multiplicities.

We can augment these divisors with a graphical structure that captures the dynamics. This object plays a central role in Chapter IV.

Definition 3.8. Let $\mathfrak{G}$ be an effective pro-divisor on $\mathfrak{X}$. The **incidence graph of** $\mathfrak{G}$ is the directed multigraph with vertex set the support of $\mathfrak{G}$, and an undirected edge $\mathfrak{g} \leftrightarrow \mathfrak{h}$ if $\mathfrak{g}$ and $\mathfrak{h}$ meet on the special fiber. If $\mathfrak{g}$ is a point with multiplicity, it will have a loop, and if points meet with multiplicity there will be multiple edges.

If we further assume that f has good reduction, and hence extends to a morphism $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$, then we can augment the incidence graph with dynamical structure:

Definition 3.9. Let $\mathfrak{G}$ be an effective pro-divisor on $\mathfrak{X}$. The **dynamical incidence graph of** $f(x)$ **on** $\mathfrak{G}$ is obtained by augmenting the incidence graph of $\mathfrak{G}$ with a directed edge $\mathfrak{g} \to \mathfrak{h}$ if $\mathfrak{f}(\mathfrak{g}) = \mathfrak{h}$.

Any dynamical incidence graph has a reduction $\tilde{\mathfrak{G}}$, obtained by identifying all vertices connected with an *undirected* edge. When f has good reduction, this is the same as the dynamical incidence graph of $\tilde{f}$.

For our purposes, there are two important dynamical incidence graphs: one on the full critical locus, and another on the grand critical locus:

Definition 3.10. Let $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be a rational map with good reduction that has at least three distinct pre-critical points, or equivalently, not conjugate to a powering map. Fix coordinates on $\mathbb{P}^1_{\mathcal{O}_K}$ and identify its generic fiber with $\mathbb{P}^1_K$ such that $\{0, 1, \infty\}$ are pre-critical points. Let $\mathfrak{C}_f$ be the full critical divisor of f, and let $\mathfrak{G}\mathfrak{C}_f$ be the the grand critical divisor of f.

Then the **(full) critical incidence graph of** f is the dynamical incidence graph on $\mathfrak{C}_f$ and the **grand critical incidence graph of** f is the dynamical incidence graph on $\mathfrak{G}\mathfrak{C}_f$.

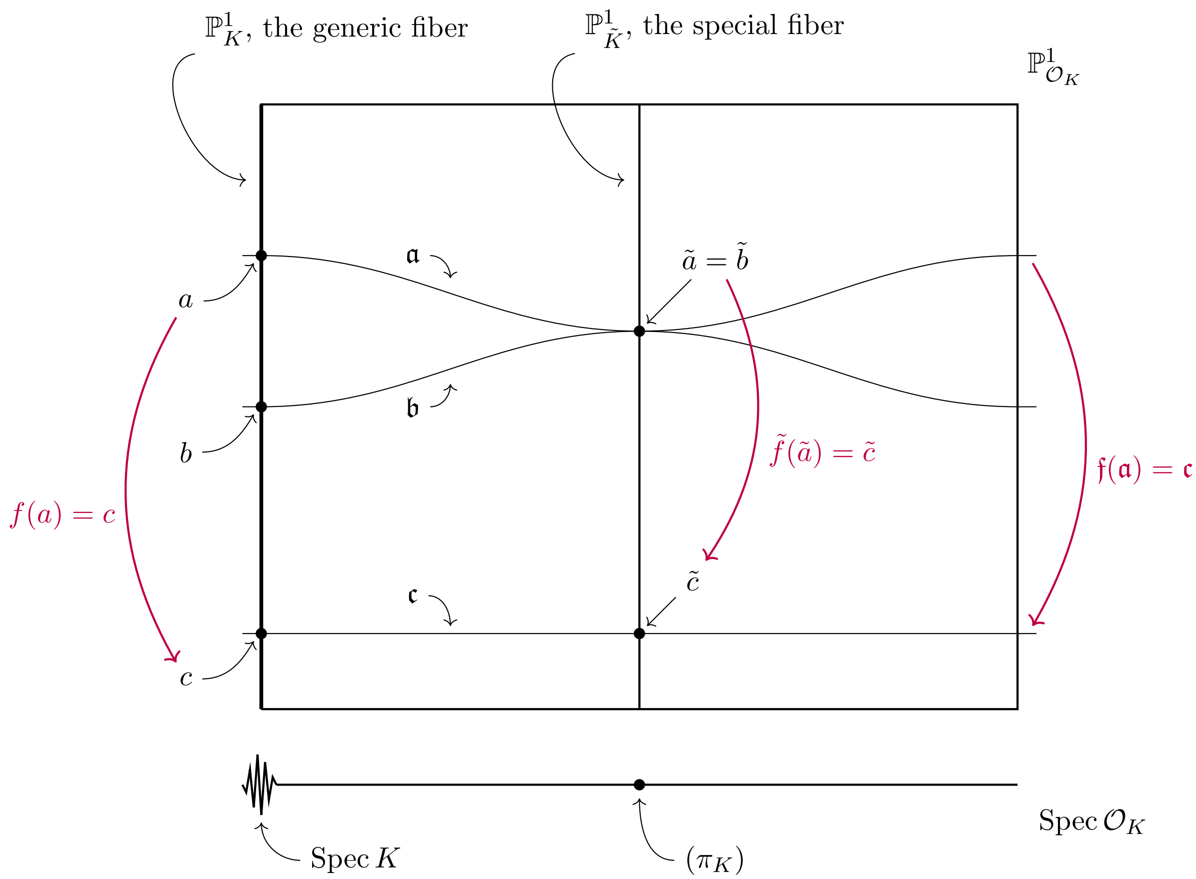


Figure 3.1: A sketch of $\mathbb{P}^1_{\mathcal{O}_K}$ and a relatively étale divisor.

There are many other pro-divisors with interesting dynamical incidence graphs, such as the portrait of a periodic point, preimages of a fixed base point, and so on. In the next chapter we will focus on the full critical pro-divisor.

Example. Suppose f has good reduction and we are given a divisor $D = a+b+c$ where $\tilde{a} = \tilde{b}$ and $f(a) = c$. We can lift this to $\mathbb{P}^1_{\mathcal{O}_K}$ to obtain a relative divisor $\mathfrak{D} = \mathfrak{a}+\mathfrak{b}+\mathfrak{c}$, where the divisors $\mathfrak{a}$ and $\mathfrak{b}$ meet on the special fiber and $\mathfrak{f}(\mathfrak{a}) = \mathfrak{c}$. The geometry of the situation is illustrated in Figure 3.1.

The intersection of $\mathfrak{a}$ and $\mathfrak{b}$ is intentionally drawn as a meeting of tangents: the divisor $\mathfrak{D}$ is *not* a relative normal crossings divisor, hence not relatively étale, and this intersection is the obstruction.

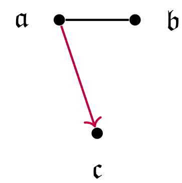


Figure 3.2: The dynamical incidence graph associated to Figure 3.1.

These intersections give rise to the covers that witness ramification in the representation on the étale fundamental group. Of course, in this situation D has only 3 points and there is a change of coordinates that would separate the two.

The dynamical incidence graph associated to Figure 3.1 is depicted in Figure 3.2.

3.4 Good Reduction II

The notion of good reduction (Definition 2.3) for dynamical systems with a divisor readily extends to pro-divisors.

Definition 3.11. Given a pro-divisor $D = \operatorname{proj\,lim} D_n$ on X, we say that it has **good reduction** if there is a proper smooth model $\mathfrak{X}$ over $\operatorname{Spec} \mathcal{O}_K$ with effective relatively étale divisors $\mathfrak{D}_n$ that restrict to X and D_n on the generic fiber.

Definition 3.12. Let $f : X \to X$ be a nearly étale dynamical system on $\mathbb{P}^1_K$. Then we say that the dynamical system has **good critical reduction** if and only if the critical pro-divisor of $f(x)$ has good reduction.

Since we only work with $X = \mathbb{P}^1_K$, the extension to $\mathfrak{X}$ is unique up to a change of coordinate over $\mathcal{O}_K$ after moving three points to $\{0, 1, \infty\}$. In this setting an effective divisor on $\mathfrak{X}$ is relatively étale if and only if it (1) has no components with multiplicity, and (2) no components which meet on the special fiber. Condition (2) originates from the stipulation that a relatively étale divisor must be a relative normal crossings divisor.

Good critical reduction is a stronger notion than usual good reduction:

Proposition 3.13. *If $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ is a rational map of degree at least two which does not have potentially good reduction, then it has bad critical reduction.*

Proof. If f has at least three distinct critical points, then make a change of coordinate so that 0, 1, and ∞ are critical points of f. Otherwise, f has two distinct critical points. If f does not have a pre-critical point distinct from these two point, then it already has bad critical reduction and there is nothing to prove. Thus we may assume f has two distinct critical points and a third distinct point which is pre-critical. Then make a change of coordinate so that 0 and ∞ are critical points, and 1 is pre-critical. These changes of coordinate may require adjoining (pre-)critical points to the ground field, after which f still has bad reduction.

Let p and q be relatively prime polynomials in $\mathcal{O}_K[x]$ such that at least one of them has a unit coefficient and $f(x) = p(x)/q(x)$. Suppose that $\tilde{p}$ and $\tilde{q}$ are both nonzero. Then when written this way, bad reduction of f is equivalent to the reductions $\tilde{p}$ and $\tilde{q}$ having a common root in the residue field, and hence p and q having roots a and b, respectively, whose reductions are equal. Since p and q have no common roots in K, we see that $f(a) = 0$ and $f(b) = \infty$. Therefore, the critical points 0 and ∞ have preimages which meet on the special fiber, so f cannot have good critical reduction.

Otherwise, exactly one of $\tilde{p}$ and $\tilde{q}$ is zero (else neither would have a unit coefficient). In this case, consider preimages of the (pre-)critical point 1, which are solutions to the equation

$$p(x) = q(x).$$

If $\tilde{p}$ is nonzero, then the above reduces to $\tilde{p}(x) = 0$, and so there is a preimage of 1 and a root of

p with the same reduction. A root of p is exactly a preimage of zero, and so this preimage appears with multiplicity on the reduction of the pre-critical divisor. If instead $\tilde{q}$ is nonzero, an essentially identical argument shows that 1 and ∞ have preimages, necessarily pre-critical, which meet on the special fiber.

$\square$

3.5 Anarboreal Representations

In the same way that anabelian representations on the étale fundamental group are "beyond" abelian (i.e. beyond cohomology), we can use our constructions and the étale fundamental group to define "anarboreal" representations which go beyond the usual arboreal representations. It is the author's hope that the analogy between abelian varieties and dynamical systems will be stronger when anarboreal representations are compared to the Tate module. Our structure has the advantage of producing a Galois action on a (typically nonabelian) group, rather than a somewhat less algebraically structured tree. In particular, when restricted to elliptic curves, our representation is exactly the usual Galois representation on the Tate module, which is not the case for arboreal representations.

The usual arboreal representation can be recovered as a "skeleton" of the anarboreal representation by forgetting most of the group structure. In fact, there is an intermediate kind of representation (the branch cycle representation of Fried [17]) which is vastly simpler than the anarboreal representation but tracks slightly more data, especially ramification-theoretic data, than the arboreal representation. We do not study the branch cycle representation here, but we hope that it will be of interest to other dynamicists.

Theorem 3.14. *Suppose that X is defined over a field K. Let $\bar{x}$ be a geometric fixed point of $f(x)$ which is not in the grand critical divisor. Then let*

$$\pi_1^{\acute{e}t}(U_\infty, \bar{x}) = \varprojlim \pi_1^{\acute{e}t}(U_n, \bar{x}),$$

which has an endomorphism $\phi : \pi_1^{\acute{e}t}(U_\infty, \bar{x}) \to \pi_1^{\acute{e}t}(U_\infty, \bar{x})$ induced by f.

Moreover, there is an étale homotopy exact sequence

$$0 \longrightarrow \pi_1^{\acute{e}t}((U_\infty)_{\bar{K}}, \bar{x}) \longrightarrow \pi_1^{\acute{e}t}(U_\infty, \bar{x}) \longrightarrow \pi_1^{\acute{e}t}(\operatorname{Spec} K, \bar{x}) \longrightarrow 0$$

which is ϕ-equivariant.

Proof. The existence of these groups and the endomorphism ϕ follows from Theorem 3.6. The exact sequence arises as the limit of the étale homotopy exact sequences associated to the U_n, as inverse limits preserve exactness in the category of profinite groups. □

When f is a rational endomorphism of $\mathbb{P}^1_K$ and not conjugate to a powering map, the grand critical locus is infinite, and so the geometric fundamental group $\bar{\Pi}$ is a free profinite group on countably many generators. The generators correspond to inertia groups over the points of the grand critical divisor – modulo a single relation, that the product of all generators in a certain order (as a limit) is the identity – and the action of ϕ is determined by its action on those points.

The exact sequence of Theorem 3.14 induces an (outer) action of Γ on $\bar{\Pi}$. As we will discuss shortly, one can recover the grand critical arboreal representation associated to $f(x)$ from this action by looking at the "skeleton" of the outer action induced by its action on conjugacy classes of inertia subgroups.

3.6 Recovering Arboreal Representations

It is straightforward to reconstruct the grand critical arboreal representation of a dynamical system from the anarboreal representation. In particular, $\pi_1^{\acute{e}t}(\bar{U}_\infty)$ is the free profinite group on countably many generators (σ_α) where σ_α generates an inertia subgroup over α. Let $\chi = \chi_K : \Gamma_K \to \hat{\mathbb{Z}}^*$ be the cyclotomic character. Then modulo conjugacy, it is well-known that $\gamma \in \Gamma_K$ acts on these generators by

$$\gamma(\sigma_\alpha) = \sigma_{\gamma(\alpha)}^{\chi(\gamma)}.$$

This is the so-called branch cycle representation [17]. It is much coarser than the full representation on the étale fundamental group, but tracks slightly more information than the arboreal representation.

It follows immediately that the grand critical arboreal representation can be identified with the action of Γ_K on conjugacy classes of inertia subgroups of $\pi_1^{\acute{e}t}(\bar{U}_\infty)$. In fact, Nakamura's anabelian weight filtration [26] allows one to identify these conjugacy classes of subgroups from just the action of Γ_K on the étale fundamental group. Many other classical results in anabelian geometry can be extended directly to anarboreal representations, such as Mochizuki's celebrated resolution of the Grothendieck conjecture for sub-p-adic fields, which already proves the Grothendieck conjecture for limits of hyperbolic curves [25]. None of these applications use the distinguished endomorphism ϕ of $\bar{\Pi}$, and it would be interesting to know if incorporating this extra information could lead to simpler proofs of these facts for dynamical anabelian representations.

Lastly, we remark that the grand critical arboreal representations play a distinguished role in this chapter and the definition of anarboreal representations. However, more general arboreal representations are important in arithmetic dynamics. For instance, Odoni's Conjecture (in various forms) predicts that arboreal representations are "very large" over number fields with only mild restrictions on the base point and rational map. These representations are not intrinsic to an unmarked dynamical system, but still quite interesting. One way to fit them into our formalism is to further puncture $\mathbb{P}^1$ at the grand orbit of the base point.

3.7 Recovering the Iterated Monodromy Representation

Anabelian ideas have already entered arithmetic dynamics by way of iterated monodromy groups, which form the focus of Pink's seminal work [30, 29, 28] on representations associated to quadratic polynomials, especially post-critically finite polynomials. These iterated monodromy groups are rather smaller than the étale fundamental groups just defined, and do not admit an endomorphism corresponding to $f(x)$, but they do capture a great deal of dynamical information.

Definition 3.15. Let $f(x) \in K(x)$ be a rational map with critical locus C. As such, $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ is a branched cover, ramified at C with branch locus $f(C)$. More generally, $f^k : \mathbb{P}^1_K \to \mathbb{P}^1_K$ is a branched cover ramified at $\bigcup_{i=0}^{k-1} f^{-i}(C)$ and with branch locus $\bigcup_{i=1}^{k} f^i(C)$. These fit into a natural tower ... $\to \mathbb{P}^1_K \xrightarrow{f} \mathbb{P}^1_K \xrightarrow{f} \mathbb{P}^1_K$.

The Galois closure of this tower is the splitting field L of all $f^k(x) - t$ over $K(t)$, and its Galois group G^{arith} is the **arithmetic iterated monodromy group**. Repeating this construction after base change to $\bar{K}$, results in G^{geom}, the **geometric iterated monodromy group**.

If we fix a point $x_0 \in \mathbb{P}^1_K$ which is not post-critical, the Galois group is determined by its monodromy action on the preimages of x_0 by $f(x)$. These preimages naturally have the structure of an infinite d-regular rooted tree T, and so G^{arith} may be identified with a subgroup of $\mathrm{Aut}(T)$.

We see immediately that these monodromy groups are a quotient of our large anarboreal representations corresponding to the union of the Galois closures of the covers $f^k : U_\infty \to U_\infty$. Equivalently, if we let $M_n = \operatorname{img} \phi^n$ and N_n the kernel of the translation action of U_∞ on the cosets of M_n, the iterated monodromy group is the quotient by $\bigcap N_n$.

Pink is primarily interested in understanding the action of Γ_K on the quotient G^{arith}/G^{geom}. This is the Galois group of the constant field of L over K. A loose interpretation of Pink's method is that it precisely identifies G^{geom} within $\operatorname{Aut}(T)$ and carefully studies conjugacy within $\operatorname{Aut}(T)$, showing that if two subgroups are element-wise conjugate, they are themselves conjugate. It is well-known that the action of Γ_K on generators factors through the cyclotomic character modulo conjugacy (the so-called branch cycle argument) but Pink's deep understanding of conjugacy within $\operatorname{Aut}(T)$ allows him to place vastly stronger restrictions on this action.

Chapter 4

The Néron-Ogg-Shafarevich Criterion

In this chapter, we use the formalism of Chapter III to apply A. Tamagawa's anabelian criterion for good reduction [40] to the good critical reduction of dynamical systems on $\mathbb{P}^1$, and use this as a starting point to give a dynamical characterization of those arboreal representations which are infinitely ramified. The content of this chapter is based on the author's preprint [37].

4.1 Good Reduction III

Tamagawa's criterion extends more or less immediately by taking a limit.

Theorem 4.1. *Let K be a p-adic field with ring of integers $\mathcal{O}_K$, residue characteristic p, and $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ a nearly étale dynamical system. Let V_∞ be $\mathbb{P}^1_K$ punctured at the critical divisor D. Then the dynamical system has good critical reduction if and only if the (outer) action of Γ on $\pi_1((V_\infty)_{\bar{K}}, \bar{x})^{(\ell)}$, the pro-$\ell$ completion of $\pi_1((V_\infty)_{\bar{K}}, \bar{x}))$ for some (all) $\ell \neq p$, is unramified.*

Proof. Extended to pro-divisors, Tamagawa's anabelian criterion for good reduction [40, p. 5.3] tells us that the Galois action on the pro-ℓ geometric fundamental group of U_∞ is unramified if and only if the dynamical system has good critical reduction, with smooth model $(\mathbb{P}^1_{\mathcal{O}_K}, \mathfrak{D})$. Since the dynamical system has good critical reduction, $f(x)$ has good reduction by Proposition 3.13 and hence extends to an endomorphism $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$. Since the map has good critical reduction, the critical pro-divisor is relatively étale and its restriction to the generic fiber is D. $\square$

Corollary 4.2. *In the notation of Theorem 4.1, the following are equivalent:*

1. *f has good critical reduction,*
2. *the dynamical incidence graph of f has no undirected edges,*
3. *the dynamical incidence graph of f has no cycles.*

Readers familiar with Tamagawa's result are likely aware that it is straightforward to prove by elementary means in the genus 0 case – one can directly construct ramified covers from a divisor which is not relatively étale. The rest of the chapter proceeds largely by following these elementary methods to the covers by $f(x)$, but while tracking more data related to the dynamics.

4.2 Stepwise Simple Reduction

Evidently, Lemma 2.12 will allow us to determine when a branch representation is infinitely ramified, and to describe the asymptotic growth of the ramification index if we can find coordinates where the rational function f admits such a power series expansion. It remains to develop some machinery to relate the ramification of a branch representation to the existence of such a coordinate.

Definition 4.3. Take a branch $(\alpha_n)_{n\in\mathbb{N}}$ for a rational map of height 0. We say that the branch has **stepwise simple reduction** if, for all n, $\tilde{\alpha}_{n+1}$ is a simple root of $\tilde{f}(x) - \tilde{\alpha}_n$. Additionally, we say that a branch has **eventually stepwise simple reduction** if it has stepwise simple reduction after removing an initial segment.

Stepwise simple reduction is a separability condition, and hence is closely related to the reduction of the critical points of $f(x)$.

Lemma 4.4. *Assume $f(x)$ has height zero, and let $(\alpha_n)_{n\in\mathbb{N}}$ be a branch which does not have stepwise simple reduction. Then there is a critical point c of $f(x)$ whose reduction $\tilde{c}$ is periodic, and the entire residual branch is residually periodic, with its entries given by the orbit of $\tilde{c}$.*

Proof. If the branch does not have stepwise simple reduction, then there are infinitely many n such that $\tilde{\alpha}_n$ is a multiple root of $\tilde{f}(x) - \tilde{\alpha}_{n-1}$, and this can occur only when $\tilde{\alpha}_n$ a critical point of $\tilde{f}$.

Thus for each such n there is a residual critical point $\tilde{c}_n$ of $f(x)$ such that $\tilde{c}_n = \tilde{\alpha}_n$. Since $f(x)$ has height zero, its derivative is nonzero and therefore $f(x)$ has only finitely many residual critical points, and each is the reduction of a critical point of $f(x)$. The sequence $(\tilde{c}_n)_{n\in\mathbb{N}}$ repeats one of those values infinitely often, and therefore that residual critical point is residually periodic, and the branch below any entry where it appears is periodic. Since the value reappears arbitrarily high in the residual branch, the whole residual branch is periodic. □

When the height h is larger than zero, the derivative of $\tilde{f}(x) = \tilde{Q}_f(x^{p^h})$ vanishes, so every point is residually critical, and hence no branch can have stepwise simple reduction. On the other hand, any choice of Q_f has height zero and hence Q_f can have branches with good critical reduction.

Observe that, residually, $\tilde{f}$ is the composition of a rational map of height zero with a power of the absolute Frobenius. So it is then natural think of $f(x)$ as being a residual twist of Q_f by Φ^h. This suggests a natural untwisting process for branches for rational maps of positive height.

Lemma 4.5. *Let $f(x)$ be a rational map of height h. Let Φ be the absolute Frobenius automorphism of the residue field. Assume that the coefficients of $\tilde{f}(x)$ are fixed by Φ^h. Then there is a correspondence between residual branches for $\tilde{f}(x)$ and branches for $\tilde{Q}_f(x)$ over the residue field:*

$$(\tilde{\alpha}_n)_{n\in\mathbb{N}} \quad \Longleftrightarrow \quad (\Phi^{-hn}(\tilde{\alpha}_n))_{n\in\mathbb{N}}$$

Proof. Since Φ^h fixes the coefficients of $\tilde{f}(x) = \tilde{Q}_f(x^{p^h})$, it commutes with $\tilde{Q}_f$. Then

$$\tilde{\alpha}_{n-1} = \tilde{f}(\tilde{\alpha}_n) = \tilde{Q}_f(\tilde{\alpha}_n^{p^h}) = \tilde{Q}_f(\Phi^h(\tilde{\alpha}_n)) = \Phi^h \tilde{Q}_f(\tilde{\alpha}_n).$$

Inductively,

$$\tilde{\alpha}_0 = \tilde{f}^n(\tilde{\alpha}_n) = \Phi^{nh}(\tilde{Q}_f(\alpha_n))$$

Therefore, $(\Phi^{-nh}(\tilde{\alpha}_n))_{n\in\mathbb{N}}$ forms a branch (over the residue field) for $\tilde{Q}_(x)$. Reversing this process turns a branch for $\tilde{Q}_f$ into a branch for $\tilde{P}(x)$. □

Adopting the convention that $\Phi^0 = \mathrm{Id}$, it is natural to incorporate the "untwisting" into Definition 4.3 and combine Lemmas 4.4 and 4.5 to treat all heights at once.

Definition 4.6. Let $f(x)$ be a rational function of height h. A branch $(\tilde{\alpha}_n)_{n\in\mathbb{N}}$ for f is said to have **stepwise simple reduction** if the corresponding "untwisted" residue branch $(\Phi^{-hn}(\tilde{\alpha}_n))_{n\in\mathbb{N}}$ for the height zero rational map $\tilde{Q}_f$ has good critical reduction in the sense of Definition 4.3.

Proposition 4.7. *Suppose $f(x)$ is a rational map with height h, so that $\tilde{f}(x) = \tilde{Q}_f(x^{p^h})$ and let $(\alpha_n)_{n\in\mathbb{N}}$ be a branch for $f(x)$. If the branch does not have eventually stepwise simple reduction, then it is residually periodic.*

Proof. If $h = 0$, this is exactly the statement of Lemma 4.4, so assume h is positive.

Consider an iterate f^k, which will have height hk, so that residually $\tilde{f}^k(x) = \tilde{R}(x^{p^{hk}})$ where $R'(x) \neq 0$. This iterate has coefficients in the same ground field as $f(x)$, so it is possible to choose a k such that Φ^{hk} fixes all the coefficients of $f(x)$.

If the branch $(\Phi^{-hn}\tilde{\alpha}_n)_{n\in\mathbb{N}}$ for $\tilde{Q}_f$ is not residually stepwise simple for $\tilde{Q}_f$, then neither is the sub-branch $(\Phi^{-hnk}\tilde{\alpha}_{nk})_{n\in\mathbb{N}}$ for $\tilde{S}(x)$. Therefore the untwisted branch is residually periodic; in other words, contained in a finite extension of the residue field. The automorphism Φ^h of this residue field has finite order, and so the twisted branch is still periodic, though possibly of larger period.

Since the subbranch is residually periodic, so too is the original branch; again the period may be larger. □

4.3 Main Results

Theorem 4.8. *Let K be a finite extension of $\mathbb{Q}_p$ and $f(x) \in K(x)$ a rational map of degree at least two. Let $\mathfrak{C}_f$ denote the full critical locus of the dynamical system (equivalently, the pro-divisor of pre-critical points). Assume that f has height zero. The following are equivalent:*

(a) The dynamical incidence graph $\mathfrak{C}_f$ has a directed cycle (i.e. a cycle with at least one directed edge).

(b) For all $\epsilon > 0$, there exists an $\alpha \in K$ and critical point c of $f(x)$ such that $|\alpha - c| < \epsilon$ and a branch over α is infinitely ramified. If c is not periodic, we can take $\alpha = c$.

(c) Some branch representation associated to $f : X \to X$ is infinitely ramified.

When (b) or (c) above occurs, the aforementioned branch $(\alpha_n)_{n\in\mathbb{N}}$ is residually periodic and there is a pre-critical branch $\{\gamma_n\}_{n\in\mathbb{N}}$ for $f(x)$ such that $|\alpha_n - \gamma_n| < 1$ for all n. Let m be the exact period of the reduction of the branch and e the ramification index of $f(x)$ on the branch; see Definition 2.9. Then for all sufficiently large n, the extension $K(\alpha_{n+m})$ over $K(\alpha_n)$ is totally ramified of degree e. In other words, the ramification index of $K(\alpha_n)$ over K grows like $Ce^{n/m}$ for some constant C.

Proof. (a)⇒(b) Suppose that $\mathfrak{C}$ has a directed cycle. Replacing $f(x)$ by an iterate, we may assume that this cycle has a single directed edge – in other words, the cycle now gives rise to a fixed point on the special fiber. Since we are working with the full critical locus, we may apply $f(x)$ to push the cycle down until it contains a critical point of $f(x)$. Without loss of generality we may assume that $f(0) = 0$ and that $\tilde{0}$ is the fixed point (i.e. our cycle) on the special fiber. If c is not periodic, let $\alpha = c$. Otherwise let α be any element of K with sufficiently high valuation. This guarantees that no branch based at α is periodic.

Now take preimages by $f(x)$ while holding the special fiber fixed, so as to obtain a branch $(\alpha_n)_{n\in\mathbb{N}}$ which does not eventually have stepwise simple reduction, because $\tilde{f}(x) - \tilde{\alpha}_n = \tilde{f}(x)$ will have a multiple root at $\tilde{0}$ for all n. Because the branch is not periodic it follows from Lemma 2.12, applied to the power series expansion of $f(x)$ around 0, that the branch is infinitely ramified. In fact, the ramification index at each step eventually grows as a power of the ramification index of $f(x)$ as a power series around 0, and hence has infinite pro-ℓ component for some prime $\ell \leq \deg f < p$.

(b)⇒(c) Trivial.

(c)⇒(a) Suppose there is some infinitely ramified branch, not necessarily critical. Then by Lemma 4.4, the restriction of this branch to the special fiber is periodic and given by the orbit of the reduction of a critical point. This residually periodic critical orbit immediately gives rise to a directed cycle in the critical incidence graph, as well as one in any arboreal representation containing the branch. This completes the proof that (c) implies (a).

Finally, in (b) and (c) we have just seen that the branch is residually periodic, and hence we can apply Lemma 2.12 to f^m, which verifies the claim about the eventual ramification behavior of the branch. □

Remark 1. One can view $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$ as a family of dynamical systems over $\mathcal{O}_K$. Theorem 4.8

says that $\mathfrak{f}$ exhibits dynamical stability if and only if the pre-critical arboreal representation and nearby representations are unramified. In other words, bifurcation is detected by ramification in the Galois action.

Given an arboreal representation whose associated dynamical incidence graph has an undirected cycle, the representation on the étale fundamental group will be ramified, but the arboreal representation need not be. However, moving the base point allows this ramification to be detected, although it will only result in a finite amount of ramification.

Theorem 4.9. *Continue to use the notation of Theorem 4.8. The following are equivalent:*

(a) The full critical dynamical incidence graph has an undirected cycle.

(b) Some branch extension $(\alpha_n)_{n\in\mathbb{N}}$ for $f(x)$ is ramified.

In fact, given any critical branch $(\gamma_n)_{n\in\mathbb{N}}$ which has a member lying in an undirected cycle of the critical incidence portrait, there is a ramified branch $(\alpha_n)_{n\in\mathbb{N}}$ such that $|\alpha_n - \gamma_n| < 1$ for all n.

Proof. Without loss of generality, we may assume that there are no directed cycles on any of the graphs in question, as otherwise Theorem 4.8 immediately verifies the claim.

(a)⇒(b) If there is an undirected cycle, then there is such a cycle with only two members, $\mathfrak{q}$ and $\mathfrak{r}$. We may assume $\mathfrak{q}$ and $\mathfrak{r}$ are preimages of the same critical point $\mathfrak{c}$. Then there are integers m and n such that $\mathfrak{f}^m(\mathfrak{q}) = \mathfrak{f}^n(\mathfrak{r}) = \mathfrak{c}$. If $m \neq n$, this would give rise to a directed cycle, contrary to assumption, so $m = n$. Taking the minimal such m, we see that $\mathfrak{f}^{m-1}(\mathfrak{q})$ and $\mathfrak{f}^{m-1}(\mathfrak{r})$ are both preimages of $\mathfrak{c}$, which meet on the special fiber. So we may replace the original initial cycle, $\mathfrak{q}$ and $\mathfrak{r}$, with $f^{m-1}(\mathfrak{q})$ and $f^{m-1}(\mathfrak{r})$. After possibly making a change of coordinates, we may assume that none of $\mathfrak{c}$, $\mathfrak{q}$, are $\mathfrak{r}$ at the point at infinity. In particular, they have counterparts c, q, and r in K with which we can perform usual arithmetic calculations.

Now let $\alpha_0 \in K$ be any point such that $v_K(\alpha_0 + f(q)) = 1$ – we have replaced the fraktur letters with their standard counterparts to emphasize that we now . There is a branch for $f(x)$ based at α_0 which is ramified. Observe that the choice of α_0 in combination with the fact that $f'(c) = 0$ guarantees that $f(x+q) - \alpha_0$ has a root with non-integer valuation: consider the Newton polygon after expanding $f(x+q) - \alpha_0$ as a power series around 0. The corresponding root of $f(x) - \alpha_0$ therefore gives rise to a nontrivial ramified extension, and any branch containing it will suffice.

(b)⇒(a) If a branch $(\alpha_n)_{n\in\mathbb{N}}$ is ramified, then the reduction of some $f(x) - \alpha_n$ has a double root. This double root must be a residual critical point of $f(x)$, and hence gives rise to an undirected edge (hence undirected cycle) in the critical dynamical incidence graph.

The final remark follows from the fact that the branches in question are all residually pre-critical, and hence entrywise close to a pre-critical branch. However, the ramified branch cannot always be made arbitrarily close to a pre-critical branch. □

Combined, Theorems 4.8 and 4.9 tell us that, in a sense, "all" of the possibilities for ramification associated to branches for $f(x)$ can be understood by looking at the pre-critical locus and branches based on annuli around it.

Theorem 4.10. *Assume that $p > d = \deg f$, in particular f has height zero. If an arboreal representation is infinitely ramified, one of its branch representations is infinitely ramified.*

Proof. Evidently if some branch is infinitely ramified, so too is the arboreal representation.

Let $\mathfrak{T}$ be the dynamical incidence portrait associated to the preimage tree. If no branch representation is infinitely ramified, then $\mathfrak{T}$ can have no directed cycles by Theorem 4.8. As such, a given branch can ramify at most $2d-2$ times, corresponding to the residual critical points lying on the branch. Therefore, the ramification index is bounded uniformly by $(d!)^{2d-2}$ across branches. This ramification is tame because we have assumed $p > d$, so by Abhyankar's lemma the ramification in all the branches can be trivialized by replacing K with any totally tamely ramified extension of degree $(d!)^{2d-2}$. Thus the full arboreal extension, obtained as the compositum of all the branch extensions, is unramified after a finite base change, and therefore only finitely ramified over K. □

Remark 2. The previous theorem holds when the ramification is wild. One must take care to show that after an unramified base change which is *independent of the branch*, some initial segment of the branch is totally ramified of bounded degree, and afterward unramified. A p-adic field has only finitely many extensions of bounded degree and so there is a single finite base change which trivializes all the possibilities for ramification.

4.4 Applications

4.4.1 Post-Critically Finite Maps

Our results are especially powerful when applied to post-critically finite maps over number fields.

Combining the main result of this paper with that of [36], we can exactly and effectively determine which primes are infinitely ramified in branch representations of of post-critically finite polynomials of prime-power degree with good reduction.

Corollary 4.11. *Suppose $f(x)$ is a post-critically finite rational map with everywhere good reduction defined over a number field K. Let Δ_f be the (necessarily finite) set $\bigcup_{f'(c)=0}\{c - f^n(c) \mid n \in \mathbb{N}_+\}$. Then a prime $\mathfrak{p}$ of K for which $f(x)$ has height zero is infinitely in ramified in some arboreal representation if and only if $\mathfrak{p}$ divides a member of Δ_f.*

In fact, an arboreal representation with base point $\alpha \in K$ is infinitely ramified at a prime $\mathfrak{p}$ of good reduction for which $f(x)$ has height zero if and only if there is a critical point c of $f(x)$ which *is periodic modulo $\mathfrak{p}$ and α is in the orbit of c modulo $\mathfrak{p}$.*

There are only finitely many primes for which $f(x)$ has nonzero height, as such a prime has to divide its degree.

Example. Let c be a root of $t^3 + 2t^2 + t + 1$, and set $f(x) = x^2 + c$. This is the post-critically finite map associated to the Douady rabbit. The critical orbit is

$$\{0, c, c^2 + c, \infty\},$$

where the first three points form a periodic cycle and the last is fixed.

This polynomial has everywhere good reduction, and only has positive height for primes lying over 2. Since all the critical points are periodic, for any prime $\mathfrak{p}$ not dividing 2 there is an arboreal representation for $f(x)$ which is infinitely ramified at p. The base point for this representation can be chosen to be $\mathfrak{p}$-adically near any periodic point of $f(x)$.

For any particular base point $\alpha \in K$ which is not periodic for $f(x)$, the arboreal representation can only ramify at one of the finitely many prime $\mathfrak{p}$ which divide one of $\alpha, \alpha - c, \alpha - c^2 - c$, or $\alpha - \infty$ where a prime $\mathfrak{p}$ divides the latter if and only if $\frac{1}{\alpha}$ is divisible by $\mathfrak{p}$.

As far as ramification at divisors of 2, this is a post-critically finite map of prime degree, so the main result of [36] implies that all such primes are infinitely ramified, and in fact deeply wildly ramified in a precise way.

To be even more concrete, if we take $\alpha = 5$ then it is straightforward to check that the arboreal representation is infinitely ramified at exactly the following primes: 2, and the ideals $(5 - \alpha)$, $(2\alpha^2 + 3\alpha + 1)$, $(\alpha^2 + 2\alpha + 3)$, which lie over 181, 7 and 19, respectively.

4.4.2 Post-Critically Infinite Maps

The pre-critical arboreal representations of post-critically infinite maps over global fields exhibit surprising behavior in contrast to abelian varieties: these arboreal representations are infinitely ramified at infinitely many primes, while the representation on the Tate module of an abelian variety can only ramify at finitely may primes.

Proposition 4.12. *Let $f(x)$ be a rational map defined over a number field K, and c a critical point of $f(x)$ whose orbit is infinite. Then the arboreal representation associated to $f(x)$ and c is ramified at infinitely many primes.*

Proof. It follows from results of Silverman [35] that there are infinitely many primes modulo which c is periodic. Restricting to primes of good reduction and where $f(x)$ has height zero, we obtain infinitely many primes $\mathfrak{q}$ such that $\tilde{c}$ is periodic. The relationship $\tilde{f}^n(\tilde{c}) = \tilde{c}$ gives rise to a directed cycle on the full critical dynamical incidence graph, and since c itself is not periodic, Theorem 4.8 guarantees that a branch over c is infinitely ramified at $\mathfrak{q}$. □

A dynamical system over $\operatorname{Spec} \mathcal{O}_K$ restricts to a family of dynamical systems over $\operatorname{Spec} S^{-1}\mathcal{O}_K$, for any finite set of primes S. As was pointed out in Remark 1, an arboreal representation ramified at p is dynamically unstable; Proposition 4.12 says that dynamical systems over number fields are very badly unstable at many primes.

Remark 3. On the other hand, it is the author's suspicion that, at least for "most" rational maps, only arboreal representations based at a point in the forward or backward orbit of a critical point can be infinitely ramified at infinitely many primes. The reason is that for such primes, not only must there be a critical point which is periodic modulo infinitely many primes, but the residual orbits of that critical point must contain the base point in their orbits as well. In other words, there must be infinitely many primes which divide an entry from both

$$\{P^n(c) - c \mid n \in \mathbb{N}\} \quad \text{and} \quad \{P^m(c) - \alpha_0 \mid m \in \mathbb{N}\}$$

Both sets are fairly sparse, and explicit computer calculations seem to indicate that they are unlikely to have many prime divisors in common.

4.4.3 Abelian Dynamical Extensions

Abelian dynamical extensions, especially over global fields, have attracted significant interest recently [4, 15]. In fact, it is known that if an arboreal representation for a rational map $f(x)$ over a number field is abelian, then $f(x)$ must be post-critically finite [15]. We can recover this by purely local methods in the special case of full critical arboreal representations: if any critical point has infinite orbit, then Proposition 4.12 furnishes infinitely many primes at which the arboreal representation is infinitely tamely ramified. But no such extension can be abelian – in fact, the associated decomposition subgroup already fails to be abelian, and even remains nonabelian after any finite base change. So not only does this arboreal representation fail to be abelian, it is quite farm from abelian: it has infinitely many infinite nonabelian decomposition groups.

Of the known examples of abelian arboreal representations, many are pre-critical, such as the Lattés maps appearing in explicit class field theory of totally imaginary quadratic fields, or the trivial case of a powering map based at zero.

Moreover, the constraints we obtain on ramification are particularly precise for PCF rational maps. In combination with global class field theory, this goes a long way towards restricting the possibilities for abelian arboreal representations.

Chapter 5

Higher Ramification and Sen's Theorem

In the previous chapter, our main results required a restriction to rational maps with height zero. For abelian varieties, this corresponds to studying T_ℓ for $\ell \neq p$. It is natural to wonder what happens when $\ell = p$. For elliptic curves, this motivated the theory of crystalline representations, and p-adic Hodge theory more generally. While we lack (as yet) a dynamical analogue for the classification furnished by p-adic Hodge theory, developing one would be extremely interesting. It is worth noting that p-adic Hodge theory itself already makes use of dynamical constructions such as the maximal cyclotomic extension or the tower of radical extension $\mathbb{Q}_p(\sqrt[\infty]{p})$ and has even been applied to questions in arithmetic dynamics [8, 39].

An important early result in the foundations of p-adic Hodge theory is Sen's theorem, which says that when a Galois extension L/K is a p-adic Lie group, the natural Lie filtration (which essentially corresponds to iterating the multiplication-by-p map) and the higher ramification filtration mutually refine each other up to change of index. In this chapter, we prove a dynamical analogue of Sen's theorem which replaces the Lie filtration with a dynamical "branch filtration". The content of this chapter is largely based on the author's paper [36].

5.1 Notation

In this chapter, we require some further notation, and will also restrict the class of rational maps we study. Through this chapter we make the following assumptions:

- $f(x) \in \mathcal{O}_K[x]$ is a polynomial,

- f has good reduction,

- $\deg f = q = p^r$,

- $f(0) = 0$,

- $f(x) = x^q = x^{p^r}$ modulo π_K,

- $\alpha_0 \in K$, the base point, has nonzero valuation.

While seemingly restrictive, we will see that all post-critically bounded polynomials can be put in this form. Even the choice of base point causes no trouble – the essence of that condition is that the base point is p-adically near a fixed point, and when $f(x) = x^q$ modulo π_K, there is always some iterate of f with a fixed point near the base point. We note that this dynamically interesting family includes all post-critically finite polynomials of prime-power degree, whose arboreal representations have attracted significant interest over global fields [1, 10].

5.2 Post-Critically Bounded Polynomials

The main results of this chapter apply to a fairly general family of polynomials, but were first motivated by computations with post-critically finite polynomials. In this section, we prove that all post-critically bounded polynomials are in this family.

After possibly extending the ground field, we will be able to make a change of coordinate such that f is monic, has integral coefficients, fixes zero, and reduces to a powering map modulo π. In fact, we will show that its coefficients satisfy certain inequalities, and this is what ensures PCB polynomials will be in the family we consider. A few other versions and proofs of Proposition 5.1 appear in the literature [2, 14, 6].

Proposition 5.1. *If a polynomial, not necessarily with good reduction, has degree $q = p^r$ and is post-critically bounded, then it has a polynomial conjugate f which is monic, integral, fixes 0, has explicit good reduction and further satisfies*

$$v(f_i) + v(i) \geq v(q) = rv(p) \quad \text{for all } 1 \leq i \leq q.$$

Proof. Let $g(x)$ be the initial polynomial. After conjugating, we may assume that $g(x)$ is monic and fixes zero; conjugates also remain post-critically bounded. This conjugation may require taking a $(p-1)$th root of the leading coefficient of $g(x)$ (at most tame) and adjoining a fixed point of g to the ground field. Call this conjugate $f(x)$. While $f(x)$ does not necessarily have integral coefficients at this point, we will show that $\frac{f'(x)}{q}$ is in $\mathcal{O}_K[x]$, from which the claimed inequality of valuations follows, and hence integrality as well.

Suppose otherwise, that $\frac{f'(x)}{q}$ is not in $\mathcal{O}_K[x]$. This guarantees a positive slope in the Newton polygon for $\frac{f'(x)}{q}$, the steepest slope of which ends at the vertex associated to the leading term. This slope must be strictly steeper than the steepest slope of the Newton polygon of $f(x)$ because every non-leading vertex moves down in passing from $f(x)$ to $\frac{f(x)}{q}$. However, this means if we take a critical point associated to this steepest slope, $v(f(c)) = qv(c) < v(c)$, hence $v(f^2(c)) < q^2 v(c)$ and so on, therefore $v(P^n(c)) \to -\infty$ and hence the critical orbit is unbounded, a contradiction. $\square$

This only tells us that a post-critically bounded polynomial has some conjugate of the desired form. Conjugation moves the base point, and a priori could leave us with a base point of valuation zero, contrary to our requirements. It turns out that, after possibly replacing $f(x)$ by an iterate, there is always a choice of conjugate such that the new base point has nonzero valuation. This is elaborated on in Section 4.2.

Eventually, we will look at $\mathcal{N}_n$ and co$\mathcal{N}_n$, the Newton polygon and copolygon associated to $f(x+\alpha_n)-\alpha_{n-1}$. When we expand this expression, the coefficients of the resulting polynomial involve binomial coefficients, and so to understand these polygons we need some control over the binomial coefficients as well. For visual simplicity we drop a set of parentheses when taking the valuation of binomial coefficients,

$$v\left(\binom{m}{n}\right) = v\binom{m}{n}.$$

Lemma 5.2. *Assume that the valuation v is normalized so that $v(p)$ is an integer. Fix a positive integers i, j, k with $j \geq i$ and $j \geq p^k$.*

(i) If $p^k \leq i < p^{k+1}$, then

$$v\binom{j}{i} \geq v\binom{j}{p^k}.$$

(ii) Additionally,

$$v\binom{j}{p^{k+1}} \geq v\binom{j}{p^k} - v(p),$$

with equality if and only if $v\binom{j}{p^k} \neq 0$.

Proof. Both claims follow from Kummer's theorem [20], which states that the p-adic valuation of a binomial coefficient $\binom{j}{i}$ is $cv(p)$, where c is the number of carries when adding i and $j-i$ in base p.

Applying that theorem, we see that a lower bound for the valuation of $\binom{j}{i}$ when the leading base p digit of i is in the ℓth place is the number of consecutive zeros in the base-p expansion of j starting at the ℓth digit. Notice that if $i = p^\ell$ then this is exact, but it can be larger in general, from carries that occur before the ℓth digit.

The condition $p^k \leq i < p^{k+1}$ says exactly that i's leading base p coefficient is in the kth place.

From these observations, (i) and the inequality of (ii) are immediate by taking $\ell = k$ and $\ell = k+1$. As to the last claim: the valuations in question are nonnegative integers, so equality is impossible if $v\binom{j}{p^k}$ is zero, and conversely if $v\binom{j}{p^k}$ is nonzero then the change from p^k to p^{k+1} loses exactly one of the aforementioned zeros in its base p expansion. $\square$

5.3 Higher Ramification and the Newton (co)Polygon

We briefly summarize some facts about the higher ramification groups and especially the Hasse-Herbrand transition function. This section is essentially a summary of Lubin's exposition of higher ramification [22], to which we refer the reader for complete proofs.

Definition 5.3. Let $g(x)$ be a Laurent series in $\mathcal{O}_K((x))$. The **Newton polygon of** $g(x)$ is defined to be the lower convex hull of the set $\{(i, v(g_i)) \mid i \in \mathbb{Z}\}$ inside $\mathbb{R}^2$. The **Newton copolygon of** $g(x)$ is the dual to the Newton polygon in the sense of convex bodies. Explicitly, the Newton copolygon can be viewed as the graph of the following function from $\mathbb{R}$ to $\mathbb{R}$:

$$T_g(u) = \min_{i \in \mathbb{Z}} \{g_i + iu\}.$$

We remark that the copolygon T_g might be more productively viewed as the tropicalization of the curve $y = g(x)$ with the infinite vertical edges deleted. The fact that tropicalization is functorial can make the Newton copolygon more convenient to work with than the Newton polygon; on the other hand, the Newton polygon is typically easier to calculate. The primary reason the graph must be considered is to count multiplicities of zeros of $g(x)$ associated to the vertices. Given a sufficiently nice theory of tropical schemes, one could work with the tropicalization of $g(x) = 0$ instead.

In this chapter, we are interested in a particular family of Newton (co)polygons.

Definition 5.4. Let (α_n) be a branch of $f(x)$. We define $\mathcal{N}_n$ and co$\mathcal{N}_n$ to be the Newton polygon and copolygon, respectively, of $f(x+\alpha_n)-\alpha_{n-1}$. Though it does not appear in the notation, these invariants depend on the branch.

Since we have already fixed a branch and will only modify it by re-indexing, no confusion should arise from leaving it out of the notation.

5.4 Asymptotic Shape of (co)$\mathcal{N}_n$

Using Lemma 2.12, we are able to take a step towards more precise information about the Newton polygons $\mathcal{N}_n$. In particular, for large n their shape essentially stabilizes up to explicitly given errors.

Lemma 5.5. *We continue to assume* $\deg f = q = p^r$. *For n sufficiently large, the Newton polygon $\mathcal{N}_n$ of $f(x+\alpha_n)-\alpha_{n-1}$ has at most $r+1$ vertices, whose x-coordinates can only be powers of p.*

Thus $\mathcal{N}_n$ is the lower convex hull of the points (p^k, y_{p^k}), where the height y_{p^k} is given by

$$y_{p^k} = \min_{p^k \le j \le q} \left\{ v\binom{j}{p^k} + v(f_j) + (j-p^k)v(\alpha_n) \right\}.$$

Proof. Let $g(x) = f(x+\alpha_n)-\alpha_{n-1}$. Expanding and collecting terms, we see that

$$g_i = \sum_{j=i}^{q} \binom{j}{i} f_j \alpha_n^{j-i}.$$

Hence

$$v(g_i) \ge \min_{i \le j \le q} \left\{ v\binom{j}{i} + v(f_j) + (j-i)v(\alpha_n) \right\}. \tag{5.1}$$

The fractional parts of the terms in the minimum, which come from $(j-p^k)v(\alpha_n)$, are all distinct so long as $0 < |v(\alpha_n)| \le \frac{1}{q}$, and from Lemma 2.12 we know this is the case for all sufficiently large

n. As such, the terms themselves are distinct and so the inequality (5.1) is actually an equality. Additionally, $v(Q_1) \neq \infty$ since the minimum in (5.1) is evidently finite.

Since $g_0 = 0$, but $g_1 \neq 0$, the Newton polygon has a vertical line through $(1, v(g_1))$. The leading coefficient is 1, so there is another vertex at $(q, 0)$.

To show that $\mathcal{N}_n$ only has vertices at prime powers, we will prove something slightly stronger: that $v(g_i)$ for i between p^k and p^{k+1} has valuation at least $v(g_{p^k}) + (p^k - i)v(\alpha_n)$, or, in other words, such points (i, Q_i) are above the line through $(p^k, v(g_{p^k}))$ with slope $-v(\alpha_n)$. Because $|v(\alpha_n)| \leq \frac{1}{q}$, the slope of that line through $(p^k, v(g_{p^k}))$ is so shallow, that this line always passes above $(q, 0)$ and so no point above this line can be a vertex *whether or not* (p^k, g_{p^k}) *is itself a vertex*. Since we will prove that every point strictly between p^k and p^{k+1} does lie above such a line, none of them can be vertices, hence the only admissible locations for vertices are at prime powers.

And so we compute, for $p^k \leq i < p^{k+1}$:

$$\begin{aligned} v(g_i) &= \min_{i \leq j \leq q} \left\{ v\binom{j}{i} + v(f_j) + (j - i)v(\alpha_n) \right\} \\ &= \min_{i \leq j \leq q} \left\{ v\binom{j}{i} + v(f_j) + (j - p^k)v(\alpha_n) \right\} + (p^k - i)v(\alpha_n) \\ &\geq \min_{p^k \leq j \leq q} \left\{ v\binom{j}{i} + v(f_j) + (j - p^k)v(\alpha_n) \right\} + (p^k - i)v(\alpha_n) \end{aligned} \tag{5.2}$$

This is nearly the desired inequality, but with $v\binom{j}{i}$ rather than $v\binom{j}{p^k}$. To resolve this issue, we apply Lemma 5.2, which tells us that if $p^k \leq i < p^{k+1}$, then

$$v\binom{j}{i} \geq v\binom{j}{p^k}.$$

Continuing where we left off at (5.2):

$$\begin{aligned} v(Q_i) &\geq \min_{p^k \leq j \leq q} \left\{ v\binom{j}{p^k} + v(f_j) + (j - p^k)v(\alpha_n) \right\} + (p^k - i)v(\alpha_n) \\ &= v(g_{p^k}) + (p^k - i)v(\alpha_n) \end{aligned}$$

as was to be shown.

Lastly, y_{p^k} is simply $v(g_{p^k})$, which is given by (5.1). □

In the preceding description of the heights of the points defining $\mathcal{N}_n$, one might notice that for sufficiently large n, the "error terms" $(j - p^k)v(\alpha_n)$ appearing in the minimum are very small. So we should expect the polygons $\mathcal{N}_n$ to be quite similar when n is large. This is the case, as we will prove shortly, although tracking these error terms make the proof less clear than we might like. While it is tempting to consider just $\lim_{n\to\infty} \mathcal{N}_n$ (as functions on $\mathbb{R}$) this shape may have fewer vertices than all of the $\mathcal{N}_n$. Carefully following the error is what allows us to establish that the number of vertices is independent of n, for n sufficiently large.

The main idea is that the height of each point defining $\mathcal{N}_n$ has a main term and an error term. Sometimes, one can identify a vertex or non-vertex simply by the position of its main term relative to the other main terms, because the error is small. When vertices are not distinguished by the main term, it must be the error term distinguishing the vertex, and there is sufficient regularity in these error terms that when a vertex appears in $\mathcal{N}_n$ due to the error term, it continues to do so for $\mathcal{N}_{n+1}$ and so on.

This important, but technical, geometric fact is made precise by the following lemma.

Lemma 5.6. *Let m, m', m'' and $0 \leq e, e', e'' \leq q-1$ be nonnegative integers, $0 \leq s < t < u \leq r$ positive integers, and $|C| \leq 1$ a constant.*

For $n \geq 2$, define the following sequences of points:

$$\mathcal{P}_n = \left(p^s, m + e\frac{C}{q^n}\right),$$
$$\mathcal{P}'_n = \left(p^t, m' + e'\frac{C}{q^n}\right),$$
$$\mathcal{P}''_n = \left(p^u, m'' + e''\frac{C}{q^n}\right).$$

Then the point $\mathcal{P}'_n$ lies below the line connecting the points $\mathcal{P}_n$ and $\mathcal{P}''_n$ if and only if the point $\mathcal{P}'_{n+1}$ lies below the line connecting the points $\mathcal{P}_{n+1}$ and $\mathcal{P}''_{n+1}$.

Proof. The key observation $(*)$ is the following: the slope of a line between any two lattice points over p^u and p^s has denominator $p^u - p^s$, which is always smaller than $q-1$, so if such a line doesn't pass through some lattice point, the closest it can approach that lattice point is at a vertical distance of $\frac{1}{q-1}$.

With that in mind, $\mathcal{P}'_n$ lies below the line connecting $\mathcal{P}_n$ and $\mathcal{P}''_n$ if and only if

$$m' + e'\frac{C}{q^n} < \frac{p^t - p^s}{p^u - p^s}\left(m + e\frac{C}{q^n}\right) + \frac{p^u - p^t}{p^u - p^s}\left(m'' + e''\frac{C}{q^n}\right). \tag{5.3}$$

Our goal is to show that (5.3) holds with $n+1$ in place of n:

$$m' + e'\frac{C}{q^{n+1}} < \frac{p^t - p^s}{p^u - p^s}\left(m + e\frac{C}{q^{n+1}}\right) + \frac{p^u - p^t}{p^u - p^s}\left(m'' + e''\frac{C}{q^{n+1}}\right). \tag{5.4}$$

We can see that inequality (5.3) roughly decomposes into two pieces: one involving only the main terms m, m', m'', and one involving just the error terms e, e', e''. This leads us to consider two cases:

$$m' \le \frac{p^t - p^s}{p^u - p^s} m + \frac{p^u - p^t}{p^u - p^s} m'' \tag{5.5}$$

and

$$m' > \frac{p^t - p^s}{p^u - p^s} m + \frac{p^u - p^t}{p^u - p^s} m''. \tag{5.6}$$

Case 1. If (5.5) holds, then subtracting it from (5.3) and dividing by q yields

$$e' \frac{C}{q^{n+1}} < \frac{p^t - p^s}{p^u - p^s} e \frac{C}{q^{n+1}} + \frac{p^u - p^t}{p^u - p^s} e'' \frac{C}{q^{n+1}}. \tag{5.7}$$

Adding (5.7) back to our assumption (5.5) yields the desired inequality (5.4). These manipulations can be reversed, so (5.5) is equivalent to (5.4) in this case.

Case 2. If (5.6) holds instead, we will have a contradiction. By our key observation ($*$), the fact that (5.6) is a strict inequality means that

$$m' - \frac{p^t - p^s}{p^u - p^s} m - \frac{p^u - p^t}{p^u - p^s} m'' \ge \frac{1}{q-1} \tag{5.8}$$

However, we can rearrange (5.3) to obtain

$$m' - \frac{p^t - p^s}{p^u - p^s} m - \frac{p^u - p^t}{p^u - p^s} m'' < -e' \frac{C}{q^n} + \frac{p^t - p^s}{p^u - p^s} e \frac{C}{q^n} + \frac{p^u - p^t}{p^u - p^s} e'' \frac{C}{q^n}. \tag{5.9}$$

The left hand side is at least $\frac{1}{q-1}$ by (5.8), but the right hand side is too small to allow this:

$$\begin{aligned} \left| -e' \frac{C}{q^n} + \frac{p^t - p^s}{p^u - p^s} e \frac{C}{q^n} + \frac{p^u - p^t}{p^u - p^s} e'' \frac{C}{q^n} \right| &= \left| -e' + \frac{p^t - p^s}{p^u - p^s} e + \frac{p^u - p^t}{p^u - p^s} e'' \right| \left| \frac{C}{q^n} \right| \\ &\le \left| \frac{p^t - p^s}{p^u - p^s} (q-1) + \frac{p^u - p^t}{p^u - p^s} (q-1) \right| \left| \frac{C}{q^n} \right| \\ &= |q-1| \left| \frac{C}{q^n} \right| \\ &\le (q-1) \frac{1}{q^2} \\ &< \frac{1}{q}. \end{aligned} \tag{5.10}$$

Together, (5.8), (5.9), and (5.10) give $\frac{1}{q-1} < \frac{1}{q}$, clearly a contradiction. □

With Lemma 5.6 in hand, we are ready to prove the final result of this section, a crucial input to our main results.

Proposition 5.7. *There is a positive integer V depending only on the polynomial $f(x)$ and the sign of $v(\alpha_0)$ such that for all n sufficiently large the Newton polygon $\mathcal{N}_n$ of $f(x+\alpha_n)-\alpha_{n-1}$ has exactly V vertices.*

In fact, there are nonnegative integers r_i, m_i, e_i, for $1 \leq i \leq V$, depending only on $f(x)$ and $v(\alpha_0)$, and a constant C which depends only on the degree q and sequence of valuations $(v(\alpha_n))_{n\in\mathbb{N}}$, such that, for all sufficiently large n, the vertices of $\mathcal{N}_n$ are all of the form

$$\left(p^{r_i}, m_i + \frac{e_i}{q^n}C\right).$$

Proof. We start by combining Lemma 2.12 of Chapter 2, which characterizes the good behavior of ramification for large n, with Lemma 5.5. Together, these lemmas tell us that there is some N such that $|v(\alpha_N)| \leq \frac{1}{q^2}$ and all the conclusions of both Lemma 2.12 and Lemma 5.5 hold for $n \geq N$. For the remainder of the proof, we only discuss $n \geq N$. Set $C = q^N v(\alpha_N)$; this is independent of our choice of N, which we can see by again applying Lemma 2.12:

$$q^n v(\alpha_n) = q^n \frac{v(\alpha_N)}{q^{n-N}} = q^N v(\alpha_N) = C, \tag{5.11}$$

from which it also follows that, for all $n \geq N$, $v(\alpha_n) = \dfrac{C}{q^n}$.

Now, recall the description of $\mathcal{N}_n$ given by Lemma 5.5: it is the lower convex hull of the points (p^k, y_{p^k}), where

$$y_{p^k} = \min_{p^k \leq j \leq q} \left\{ v\binom{j}{p^k} + v(f_j) + (j-p^k)v(\alpha_n) \right\}.$$

Since $|v(\alpha_n)| \leq \frac{1}{q^2}$ and $|j-p^k| \leq q-1$,

$$|(j-p^k)v(\alpha_n)| < 1,$$

while $v\binom{j}{p^k} + v(f_j)$ is an integer. Moreover, all the terms $(j - p^k)v(\alpha_n)$, for k fixed and n, j varying, have the same sign, and so the index j which achieves the minimum is determined entirely by the "main term" $v\binom{j}{p^k} + v(f_j)$ except when ties must be broken. The ties always break the same way, and depend only on the sign of $v(\alpha_0)$: in the integral case, one takes the smallest index j achieving the tie value, while in the non-integral case one takes the largest such index. These are the choices which minimize the expression when there is a tie for the larger contribution of $v\binom{j}{p^k} + v(f_j)$.

So for each k, the height of the point above p^k is

$$y_{p^k} = \min_{p^k \leq j \leq q} \left\{ v\binom{j}{p^k} + v(f_j) + (j - p^k)v(\alpha_n) \right\}$$

with the minimum achieved by a unique index j between p^k and q (inclusive). Then define

$$M_{p^k} = v\binom{j}{p^k} + v(f_j),$$

$$E_{p^k} = j - p^k.$$

The above argument shows that j is independent of n, and hence M_{p^k} and E_{p^k} are also independent of n. It is also clear that these constants are all positive. Moreover, because $v(\alpha_n) = \frac{C}{q^n}$, we see that

$$y_{p^k} = M_{p^k} + \frac{E_{p^k}}{q^n} C. \tag{5.12}$$

From (5.12), what remains to be shown is that the number of vertices and the x-coordinates of the vertices do not depend on n. This follows by induction from Lemma 5.6, which shows that that if the Newton polygon $\mathcal{N}_n$ has a vertex over p^t then the Newton polygon $\mathcal{N}_{n+1}$ does too, and conversely that if $\mathcal{N}_n$ has no vertex over p^t, then neither does $\mathcal{N}_{n+1}$. Thus $\mathcal{N}_n$ and $\mathcal{N}_{n+1}$ and so on all have the same number of vertices.

We know that $\mathcal{N}_n$ has a vertex over p^t if and only if for all s and u such that $s < t < u$ the point over p^t lies below the line segment connecting the vertices over p^s and p^u. If we let

$$m = M_{p^s},\ m' = M_{p^t},\ m'' = M_{p^u},\ e = E_{p^s},\ e' = E_{p^t},\ e'' = E_{p^u},$$

then we are in exactly the situation to which Lemma 5.6 applies: by (5.12) the points $\mathcal{P}_n, \mathcal{P}'_n, \mathcal{P}''_n$ are the points over p^s, p^t, and p^u defining $\mathcal{N}_n$, while $\mathcal{P}_{n+1}, \mathcal{P}'_{n+1}, \mathcal{P}''_{n+1}$ are the points over p^s, p^t, and p^u that are used to define $\mathcal{N}_{n+1}$. So the lemma tells us that $\mathcal{N}_n$ has a vertex over p^t if and only if $\mathcal{N}_{n+1}$ also has a vertex over p^t.

Thus, by induction, all of the vertices lie over the same x-coordinates for all $n \geq N$, and hence their number, which we call V, is constant. We let r_i be the exponents of the prime powers which appear as x-coordinates; m_i be the associated main term $M_{p^{v_i}}$; e_i the associated error coefficient $E_{p^{v_i}}$. The arguments above show that these do not depend on the choice of branch, only the valuations of the coefficients of $f(x)$ and the sign of $v(\alpha_0)$. We note that the subscripts indexing m_i and e_i are incompatible with the subscripts indexing M_{p^k} and E_{p^k}.

To conclude, we let

$$C = \lim_{n\to\infty} q^n v(\alpha_n).$$

As was shown in (5.11), the sequence $q^n v(\alpha_n)$ is eventually constant, so this limit exists; clearly it only depends on q and the sequence of valuations $\{v(\alpha_n)\}_{n\in\mathbb{N}}$. The proof above shows that C plays the desired role in defining the heights of the vertices. □

Definition 5.8. In the notation of the preceding proposition, we define the **limiting ramification**

data associated to $f(x)$ and the branch:

$$V(P,(\alpha_n)_{n\in N}) = \text{the number of vertices } V,$$

$$R(P,(\alpha_n)_{n\in N}) = (r_1, ..., r_V),$$

$$M(P,(\alpha_n)_{n\in N}) = (m_1, ..., m_V),$$

$$E(P,(\alpha_n)_{n\in N}) = (e_1, ..., e_V),$$

$$C(P,(\alpha_n)_{n\in N}) = \text{the constant } C.$$

We refer to these quantities as the "number of vertices", "vertex exponents", "main terms", "error factors", and "error coefficient", respectively.

Since the first vertex is over 1 and the last vertex is $(q, 0)$, defined by a minimum with just one term, we see that $r_1 = 0$ and $r_V = r$ and $m_V = e_V = 0$.

As was pointed out in Proposition 5.7, V, R, M, and E, only depend on the (ordered) valuations of the coefficients of $f(x)$ and the sign of $v(\alpha_0)$, while C only depends on the degree q of $f(x)$ and the sequence $(v(\alpha_n))_{n\in N})$ of the members of the branch. The calculation of these parameters is effective, and algorithms for their computation are outlined in Section 4, along with an example.

In fact, the only ineffective step in our results occurs in Lemma 2.12 – the proof of (b) and (c) does not give an effective determination of "sufficiently large". There are some cases where this can be circumvented; for instance, if $v(\alpha_0) = 1$ then it is straightforward to see that, for all n, $f(x) - \alpha_n$ is Eisenstein, which implies (b) and (c) hold for all n. More generally, it follows from our proof of Lemma 2.12 that if there is some N such that $v(\alpha_N)$ is not divisible by p and has smaller valuation than any coefficient of $f(x)$, then (b) and (c) hold for all $n \geq N$.

One can see quite readily from Proposition 5.7 that the polygons $\mathcal{N}_n$ have a pointwise limit (viewing them as functions on $\mathbb{R}_{\geq}0$). Some of what follows can be described in terms of that limiting polygon, and at times more simply – for instance, one could avoid using Lemma 5.6. However, valuable information is lost when working with this limit polygon: it may have fewer vertices than the actual Newton polygons $\mathcal{N}_n$ (this occurs when main terms of vertices, (p^{r_i}, m_i), are collinear). The number of vertices V is extremely important for our main result and applications, because $V-1$ is the slope of the linear change of index in our main result. Additionally, it is appealing to have such an exact description of $\mathcal{N}_n$.

5.5 Dynamical Sen

In this section, we prove the main result of this chapter, explicitly describing the higher ramification filtration of certain branch extensions, possibly after extending the ground field and adjusting the index. Such an extension may be required because the dynamics of the ramification can take some time to stabilize. As we will see shortly, the previous section amounted to showing that the ramification actually does stabilize. So for technical simplicity, we will now introduce some assumptions to the effect that we have already reached the region of stable behavior (in other words, that the results of the preceding section hold immediately for f and α_0, without first replacing the base point by some α_N). At the end we will explicitly work out the reduction of the general case to the stable case, and the adjustments required.

Besides this, it is also necessary to introduce a "tameness" assumption, that d is not divisible by p. Recall that $d = \lim_{n\to\infty} d_n$ is the eventual valuation of α_n with respect to a valuation that sends π_n to 1, and that this limit exists was shown in Lemma 2.12(c). In what follows, we will want to take a dth root of a certain unit u_n inside K_n. Recall that the unit u_n was defined by $\alpha_n = u_n \pi_n^{d_n}$, and so the presence of this dth root allows us to take a different choice of uniformizer

π_n, such that $\alpha_n = \pi_n^d$. This dth root is not necessarily in K_n, but if p does not divide d, then we can obtain a dth root of u_n after an unramified extension of K_n, which does not change the ramification along the branch. However, if p divides d then the dth root of u_n might only appear in a ramified extension of K_n, and this extra ramification interferes with our ability to extract information about ramification prior to including the dth root. We hope that this restriction can be relaxed in some or all cases – the study of some special cases suggests that if $d = d_0 p^m$ where $p \nmid d_0$, then our results still hold with d_0 in place of d. An unfortunate downside of this restriction is that it means our results are not base-change invariant – if we replace K by an extension with ramification index divisible by p and linearly disjoint from K_∞, then p is guaranteed divide d. Luckily, we at least have invariance under *tame* base change.

This leads us to introduce the following property:

Definition 5.9. A pair (f, α_0) satisfies is said to be **(upper) ramification-stable** if it satisfies the *conclusions* of Lemma 2.12 and Proposition 5.7 for all n, without the qualification "for sufficiently large n". If also $p \nmid d$, we describe the pair as **tamely** ramification-stable.

And so Lemma 2.12 and Proposition 5.7 tell us that even if f and α_0 are not ramification stable, there is some N such that f and α_N are. In the Galois case, this is equivalent to replacing an (infinite) profinite group with a finite-index subgroup which, hopefully, retains a lot of information about the original group.

Given these assumptions, our next goal is to compute the Hasse-Herbrand function of the extension K_∞/K and verify that it is arithmetically profinite, for tamely ramification-stable pairs. We will break up the computation of the Hasse-Herbrand function of K_∞/K into calculating the Hasse-Herbrand functions for the intermediate extensions K_n/K_{n-1}, composing those functions to obtain the Hasse-Herbrand function of K_n/K, and then pass to the limit. As mentioned in the introduction, we avoid assuming any of our extensions are Galois (indeed, one would expect this to be rare in general) so to study higher ramification, we employ the techniques explained by Lubin [22]. The reader is advised to take some care in passing between this and other sources (such as Serre [34]) since the ramification groups may be numbered differently; we adopt Lubin's convention.

For convenience, we remind the reader of two important polygons: the *level* n *Newton polygon* $\mathcal{N}_n$ and its dual, the *level* n *Newton copolygon* denoted co$\mathcal{N}_n$. The former is the Newton polygon of $f(x+\alpha_n)-\alpha_{n-1}$, while the latter is its dual, meaning that co$\mathcal{N}_n$ has a vertex whose x-coordinate is the negative of that slope, and the slopes of co$\mathcal{N}_n$ are the x-coordinates p^{r_i} of vertices of $\mathcal{N}_n$, in decreasing order. As such, the copolygon co$\mathcal{N}_n$ has one fewer vertex than the polygon $\mathcal{N}_n$. Ramification-stability amounts to the following explicit description of $\mathcal{N}_n$: the Newton polygon $\mathcal{N}_n$ is the lower convex hull of the following points determined by the limiting ramification data:

$$\left(p^{r_i}, m_i + e_i\frac{C}{q^n}\right) \qquad 1 \le i \le V.$$

Proposition 5.10. *Suppose the pair* (f, α_0) *is tamely ramification-stable. Then the graph of the Hasse-Herbrand transition function* ϕ_n *for* K_n/K_{n-1} *relative to* K *can be obtained by applying the following three transformations to the copolygon co*$\mathcal{N}_n$*:*

(1) Increment the x*-coordinates of each vertex by* $\operatorname{sgn}(v(\alpha_0))(d-1)v(\pi_n)$*, while modifying the* y*-coordinates to preserve the slopes of the segments between them.*

(2) Stretch horizontally by a factor of $e_{K/E}q^n$*.*

(3) Stretch vertically by a factor of $e_{K/E}q^{n-1}$*.*

The first slope of ϕ_n is 1 *and the last slope of ϕ_n is* $1/q$. *The x-coordinates of the first and last vertices of ϕ_n, are respectively,*

$$-e_{K/E}q^n(\text{shallowest slope of } \mathcal{N}_n) + \text{sgn}(v(\alpha_0))(d-1)v(\alpha_0)$$

and

$$-e_{K/E}q^n(\text{steepest slope of } \mathcal{N}_n) + \text{sgn}(v(\alpha_0))(d-1)v(\alpha_0).$$

Proof. We will prove the proposition in full for the integral case, where $v(\alpha_0) > 0$ and hence $d \geq 1$, and at the end indicate the minor modifications necessary for the non-integral case.

Let $P(x)$ be the minimal polynomial for π_n over K_{n-1}. The Hasse-Herbrand function for K_n/K_{n-1} can be obtained by applying stretches (2) and (3) to the Newton copolygon of $P(x+\pi_n)$ [22, Definition 5]. In Lubin's notation, we are taking $K = K_n$, $k = K_{n-1}$, and $k_0 = E$, and $\Psi_{v,F}$ is the copolygon of $P(x+\pi_n)$; the claimed scaling factors are obtained by expanding $e_{K/k_0} = e_{K_n/K}e_{K/E} = q^n e_{K/E}$ and similarly $e_{k/k_0} = q^{n-1}e_{K/E}$. So we only need to show that the copolygon of $P(x+\pi_n)$ can itself be obtained by applying (1) to co$\mathcal{N}_n$.

In terms of Newton polygons, (1) is equivalent to *decreasing* all of the slopes of $\mathcal{N}_n$ by $(d-1)v(\pi_n)$ (there is a sign change in the duality between polygon and copolygon!). The Newton polygons of $P(x+\pi_n)$ and $f(x+\alpha_n)-\alpha_{n-1}$ encode the valuations of the roots of the corresponding polynomials. These roots are of the form $\pi_n^\sigma - \pi_n$ and $\alpha_n^\sigma - \alpha_n$, respectively, for $\sigma \in \Gamma_K$, and our task is to relate their valuations.

In the integral case, we want to show that, for all $\sigma \in \Gamma_K$,

$$v(\pi_n^\sigma - \pi_n) = v(\alpha_n^\sigma - \alpha_n) - (d-1)v(\pi_n).$$

Recall that we selected uniformizers π_n and units u_n such that $\alpha_n = u_n\pi_n^{d_n}$. Since the pair is ramification-stable, $d_n = d$ does not vary with n, and we also assumed it is not divisible by p.

As such, u_n admits a dth root after at most an unramified extension; the transition function is insensitive to base change by unramified extensions. In other words, extending K_{n-1} and K_n by their unramified extension of degree d guarantees the presence of $\sqrt[d]{u_n}$ in our field without affecting the transition function. So altering our choice of π_n, we may write $\alpha_n = \pi_n^d$. This allows us to directly compare the valuations:

$$\alpha_n^\sigma - \alpha_n = (\pi_n^\sigma)^d - \pi_n^d \tag{5.13}$$

$$= \prod_{\zeta^d=1} (\pi_n^\sigma - \zeta\pi_n) \tag{5.14}$$

Of the terms in the product (5.14), we are only interested in $v(\pi_n^\sigma - \pi_n)$. To manage the others, notice that

$$v(\pi_n^\sigma - \zeta\pi_n) = v(\pi_n) + v\left(\frac{\pi_n^\sigma}{\pi_n} - \zeta\right). \tag{5.15}$$

If $v\left(\frac{\pi_n^\sigma}{\pi_n} - \zeta\right)$ is positive, then $\frac{\pi_n^\sigma}{\pi_n}$ is necessarily a dth root of unity modulo π_n. On the other hand, the norm from K_n to K of $\frac{\pi_n^\sigma}{\pi_n}$ is just 1; but viewed in the residue field, the norm is just the qth power. Therefore, in the residue field, $\frac{\pi_n^\sigma}{\pi_n}$ is both a dth root of unity and a qth root of unity. Because $p \nmid d$, this is only possible if $\zeta = 1$. In all other cases, $v\left(\frac{\pi_n^\sigma}{\pi_n} - \zeta\right) = 0$. Thus, (5.15) simplifies to just $v(\pi_n)$ whenever $\zeta \neq 1$, and so the valuation of (5.14) becomes

$$v(\alpha_n^\sigma - \alpha_n) = v(\pi_n^\sigma - \pi_n) + (d-1)v(\pi_n)$$

or equivalently

$$v(\pi_n^\sigma - \pi_n) = v(\alpha_n^\sigma - \alpha_n) - (d-1)v(\pi_n),$$

which is exactly the statement to which we reduced the main part of this proposition for the integral case.

For the non-integral case, i.e. when d is negative, we must replace (5.13) with

$$\frac{1}{\alpha_n^\sigma} - \frac{1}{\alpha_n} = (\pi_n^\sigma)^{|d|} - \pi_n^{|d|}.$$

The left hand side of the modified (5.13) can be written as

$$\frac{\alpha_n - \alpha_n^\sigma}{\alpha_n \alpha_n^\sigma}$$

which has valuation

$$v(\alpha_n - \alpha_n^\sigma) - 2v(\alpha_n).$$

Recall that $v(\alpha_n) = dv(\pi_n)$. Making these adjustments to (5.13) and rearranging to move the $2dv(\pi_n)$ to the right hand side, the remainder of the argument proceeds essentially unchanged until the end, where incorporating the extra $2dv(\pi_n)$ gives rise to the $\mathrm{sgn}(v(\alpha_0))$ in the statement of the proposition.

Finally, by inspecting the transformation of $\mathrm{co}\mathcal{N}_n$ into ϕ_n, one can see that the first and last slopes of ϕ_n are $\frac{e_{K/E}q^{n-1}}{e_{K/E}q^n} = \frac{1}{q}$ multiplied by the first and last slopes of $\mathrm{co}\mathcal{N}_n$. The first and last slopes of $\mathrm{co}\mathcal{N}_n$ are the first and last x-coordinates of vertices of $\mathcal{N}_n$, which are 1 and q, so together we see that the first and last slopes of ϕ_n are 1 and $\frac{1}{q}$, as claimed. Likewise, the x-coordinates can be obtained from the duality of $\mathrm{co}\mathcal{N}_n$, which turns negative slopes of $\mathcal{N}_n$ into x-coordinates of vertices, then modified according to the first two transformations. □

Remark 4. We use the assumption $p \nmid d$ in two places: to take a dth root of u_n after a merely unramified extension, and to control $v(\pi_n - \zeta\pi_n)$ by using the fact that the dth roots of unity are distinct modulo p. The former seems to be the greater obstacle to characterizing ramification in the general case.

The essence of Proposition 5.10 is that the ramification-theoretic properties of the "small" intermediate extensions K_{n+1}/K_n in the branch are somewhat stable. Neglecting scaling, all the Hasse-Herbrand functions look like a small shift of co$\mathcal{N}_n$, and the copolygon itself changes little as a function of n, in a way which is described very precisely by Proposition 5.7.

The most difficult step would appear to be composing the intermediate Hasse-Herbrand functions $\phi_1, \phi_2, ..., \phi_n$ to obtain the Hasse-Herbrand function Φ_n for K_n/K. However, this is straightforward if we can ensure that the ϕ_n's behave sufficiently well. Since ϕ_n is the identity along its first segment, one might hope that the domain on which it is the identity includes all of the vertices of the previous Φ_{n-1}.

Unfortunately, this is too much to expect in general, but we can give a sufficient condition for these functions to have sufficiently large identity segments. We can show that both post-critically bounded polynomials (of the appropriate form) and polynomials of prime degree exhibit this good behavior with respect to composition of the above Hasse-Herbrand functions.

Proposition 5.11. *Suppose (f, α_0) is tamely ramification stable.*

If the limiting Newton polygon has just one slope, or if

$$-q\frac{m_V - m_{V-1}}{p^{r_V} - p^{r_{V-1}}} > -\frac{m_2 - m_1}{p^{r_2} - p^{r_1}} + \frac{2}{p-1}|v(\alpha_0)|,$$

then for all $n \geq 2$, the leftmost vertex of ϕ_n has strictly larger x-coordinate than that of the rightmost vertex of ϕ_{n-1}

The various m_i and r_i are the quantities given by the limiting ramification data of Definition 5.8.

Proof. By the final statement of Proposition 5.10, we can rewrite the claim about the x-coordinates of those vertices in terms of the slopes of $\mathcal{N}_n$ and $\mathcal{N}_{n-1}$. We want

$$-e_{K/E}q^n(\text{shallowest slope of } \mathcal{N}_n) + \text{sgn}(v(\alpha_0))(d-1)v(\alpha_0)$$

to be strictly larger than

$$-e_{K/E}q^{n-1}(\text{steepest slope of } \mathcal{N}_{n-1}) + \operatorname{sgn}(v(\alpha_0))(d-1)v(\alpha_0).$$

For convenience, let's name the negatives of these slopes: let

$$s = -(\text{shallowest slope of } \mathcal{N}_n) \qquad s' = -(\text{steepest slope of } \mathcal{N}_{n-1}).$$

Now we can simplify and rewrite the target inequality as

$$qs > s'.$$

When there is just one slope, $s = s'$ and the inequality obviously holds. Otherwise, there are two slopes.

Now, the height of the vertex over p^{r_i} is given by $m_i + e_i \frac{v(\alpha_0)}{q^n}$. So in expressing the slopes of the segments between our vertices of interest in these terms, the quantities

$$t = -\frac{m_V - m_{V-1}}{p^{r_V} - p^{r_{V-1}}} \quad \text{and} \quad t' = -\frac{m_2 - m_1}{p^{r_2} - p^{r_1}}$$

in the statement of the proposition are the (negative) contributions of the "main terms" to the slopes s and s'. In light of this interpretation, we can write

$$s - t = \frac{e_{V-1} - e_V}{p^{r_B} - p^{r_{V-1}}} \frac{v(\alpha_0)}{q^n},$$
$$s' - t' = \frac{e_1 - e_2}{p^{r_2} - p^{r_1}} \frac{v(\alpha_0)}{q^{n-1}}.$$

As was remarked previously, $r_1 = 0$, $r_V = r$ and $e_V = 0$, because the first vertex lies over 1, while the last vertex is $(q, 0)$.

To summarize, the hypothesis of the proposition is

$$qt > t' + \frac{2}{p-1}v(\alpha_0),$$

and we have some s, s' such that

$$s - t = \frac{e_{V-1}}{q - p^{r_{V-1}}} \frac{v(\alpha_0)}{q^n},$$
$$s' - t' = \frac{e_1 - e_2}{p^{r_2} - 1} \frac{v(\alpha_0)}{q^{n-1}},$$

and our goal is

$$qs > s'.$$

But then it is enough for our two errors $q(s-t)$ and $s'-t'$ to be small enough that their sum is less than $\frac{2}{p-1}|v(\alpha_0)|$ in absolute value, as then adding these error terms to the inequality we initially assumed will preserve the inequality up to the loss of margin of error, $\frac{2}{p-1}|v(\alpha_0)|$, that we allowed ourselves. To prove that the sum of $q(s-t)$ and $s'-t'$ is small enough, it suffices to show that each is at most $\frac{|v(\alpha_0)|}{p-1}$. And indeed:

$$\begin{aligned}
|q(s-t)| &= q\frac{e_{V-1}}{q - p^{r_{V-1}}} \frac{|v(\alpha_0)|}{q^n} \\
&\leq q\frac{q-1}{q - p^{r-1}} \frac{|v(\alpha_0)|}{q^n} \\
&= \left(1 - \frac{1}{q}\right) \cdot \frac{1}{q^{n-2}} \cdot \frac{1}{p^{r-1}} \cdot \frac{|v(\alpha_0)|}{p-1} \\
&\leq \frac{|v(\alpha_0)|}{p-1},
\end{aligned}$$

and

$$\begin{aligned}
|s' - t'| &= \frac{|e_1 - e_2|}{p^{v_2} - 1} \frac{|v(\alpha_0)|}{q^{n-1}} \\
&\leq \frac{q-1}{p-1} \frac{|v(\alpha_0)|}{q^{n-1}} \\
&< \frac{q-1}{q^{n-1}} \frac{|v(\alpha_0)|}{p-1s} \\
&< \frac{|v(\alpha_0)|}{p-1},
\end{aligned}$$

where, on the second line, we use $|e_1 - e_2| \leq q-1$ rather than $\leq 2(q-1)$ because we know that e_1 and e_2 are both nonnegative. Both inequalities also require that $n \geq 2$ so that $\frac{1}{q^{n-2}}$ is at most 1. □

Corollary 5.12. *Assume the pair is tamely ramification-stable. If $f(x)$ has degree $q = p$, then it satisfies Proposition 5.11.*

Proof. Immediate, as in this case the limiting Newton polygon only has vertices over 1 and p, hence it has just a single slope. □

Corollary 5.13. *Continue to assume tame ramification-stability. If f is post-critically bounded and*

$$|v(\alpha_0)| < \frac{p-1}{2} v(p),$$

then the pair (f, α_0) satisfies the hypotheses of Proposition 5.11.

Proof. We will verify directly that Proposition 5.11 applies. If the limiting Newton polygon has just one slope, we are done. Otherwise, assume it has at least two. Then we want to verify that the following inequality holds:

$$-q\frac{m_V - m_{V-1}}{p^{r_V} - p^{r_{V-1}}} > -\frac{m_2 - m_1}{p^{r_2} - p^{r_1}} + \frac{2}{p-1}|v(\alpha_0)|. \tag{5.16}$$

Recall Proposition 5.1, which says that $\frac{f'(x)}{q}$ has integral coefficients. The first vertex of $\mathcal{N}_n$ is $(1, v(f'(\alpha_n)))$, and so its height is at least $v(q) = rv(p)$. Moreover, from Lemma 5.2(ii), we know that the height drop between vertices over p^s and p^u is at most $(u-s)v(p)$; in our notation,

$$-(m_i - m_j) \leq (r_i - r_j)v(p) \quad \text{for } i \geq j. \tag{5.17}$$

Recall as well that $r_1 = 0$ and $r_V = r$.

Working with the right hand side of (5.16),

$$\begin{aligned} -\frac{m_2 - m_1}{p^{r_2} - p^{r_1}} + \frac{2}{p-1}|v(\alpha_0)| &= -\frac{m_2 - m_1}{p^{r_2} - 1} + \frac{2}{p-1}|v(\alpha_0)| \\ &\leq \frac{r_2 v(p)}{p^{r_2} - 1} + \frac{2}{p-1}|v(\alpha_0)| && \text{apply (5.17)} \\ &< \frac{v(p)}{p-1} + \frac{2}{p-1}|v(\alpha_0)| \\ &< \frac{v(p)}{p-1} + v(p) && \text{apply the assumption} |v(\alpha_0)| < \frac{p-1}{2}v(p) \\ &= \frac{p}{p-1}v(p). && (5.18) \end{aligned}$$

Now, let us treat the left hand side of (5.16). Recall that $r_V = r$ and $m_V = 0$, so

$$-(m_V - m_{V-1}) = m_{V-1}.$$

At the same time, the first vertex has height at least p^r and the decrease in height from the first vertex to the $V-1$st vertex is at most $(r_{V-1} - 1)v(p) \geq r_{V-1}v(p)$, and therefore

$$m_V \geq (r - r_{V-1})v(p)$$

Applying this to the left hand side of (5.16), we see:

$$\begin{aligned} -q\frac{m_V - m_{V-1}}{p^{r_V} - p^{r_{V-1}}} &\geq p^r \frac{(r - r_{V-1})v(p)}{p^r - p^{r_{V-1}}} \\ &\geq \frac{r - r_{V-1}}{1 - p^{r_{V-1} - r}} v(p) && (5.19) \\ &\geq \frac{p}{p-1}v(p). && (5.20) \end{aligned}$$

Going from (5.19) to (5.20) is slightly tricky; if $r_{V-1} = r - 1$ then the two are equal, while if $r_{V-1} < 1$ then in the fraction term of (5.19), the numerator $(r - r_{V-1})$ is at least 2 and the denominator $(1 - p^{r_{V-1} - r})$ is at most 1, hence the whole quantity is at least $2v(p) \geq \frac{p}{p-1}v(p)$ (sharp when $p = 2$).

Combining (5.18) and (5.20) yields the desired inequality (5.16).

□

Remark 5. Notably, when $p \geq 5$, the inequality in Proposition 5.11 is always satisfied for $v(\alpha_0) = 1$.

It still remains to compose our Hasse-Herbrand functions. The conclusion of Proposition 5.11 describes the "good behavior" that we want in order for the Hasse-Herbrand functions to compose well: the first vertex of ϕ_n should have larger x-coordinate than the last vertex of ϕ_{n-1}. When this happens, the higher ramification behavior of the branch is quite well-controlled and highly regular. Explicit examples show that this happens in many situations besides those described by Proposition 5.11 or Corollaries 5.12 and 5.13. This leads us to introduce the following definition:

Definition 5.14. A tamely ramification-stable branch associated to f and α_0 over K is said to be **strictly tamely ramification-stable** if it also satisfies the conclusions of Proposition 5.10 and 5.11. Namely, the Hasse-Herbrand functions of the intermediate extensions K_n/K_{n-1} should have the shape specified by the conclusion of Proposition 5.10 and the x-coordinates of the first and last vertices of co$\mathcal{N}_n$ and co$\mathcal{N}_{n-1}$ should be positioned, relative to each other, according to the conclusion of 5.11.

A branch is said to be **potentially strictly tamely ramification-stable** if there is some N such that upon replacing K by K_N and re-indexing the branch to be based at α_N it is tamely ramification-stable.

Remark 6. In our definition, "tamely" refers to the restriction that $p \nmid d$. We expect that even if $p|d$, such branch extensions would still exhibit this kind of ramification stability. However, the precise expressions given in Proposition 5.10, particularly the $(d-1)v(\pi_n)$ term, may not correctly describe these cases.

Proposition 5.15. *Suppose that $p \nmid d$. If $f(x)$ has prime degree or is post-critically bounded, then any branch associated to $f(x)$ is potentially strictly tamely ramification-stable.*

Proof. Lemma 2.12 and Proposition 5.7 ensure that for all sufficiently large N, the pair is ramification-stable when K is replaced by K_N and the branch is modified to start at α_N.

For polynomials of prime degree and post-critically bounded polynomials, Corollaries 5.12 and 5.13, respectively, prove that any branch satisfies the conclusion of Proposition 5.11 after possibly increasing N, hence is strictly ramification-stable. We have already assumed $p \nmid d$, from which tameness follows. □

From the proof of Proposition 5.11, we know that if $p \nmid d$, a branch is potentially strictly tamely ramification stable when, roughly, the first (steepest) slope of $\mathcal{N}_{n-1}$ is not more than q times steeper than the last (shallowest) slope of $\mathcal{N}_n$. This property depends only on $f(x)$, not the branch. For it to fail, the first vertex of $\mathcal{N}_n$ must be relatively high compared to the others, which seems unlikely based on the structure of the minima that describe the heights of these vertices.

Before proceeding, recall the following definition:

Definition 5.16 ([22])**.** The **altitude** of an extension E/K with transition function $\Psi(x)$ is the height of the rightmost vertex of $\Psi(x)$; at times we may abbreviate this as the altitude of $\Psi(x)$.

Proposition 5.17. *Suppose our branch, associated to (f, α_0), is strictly tamely ramification-stable over K. Let V be the number of vertices from the limiting ramification data.*

Then the Hasse-Herbrand transition function $\Phi_n(x)$ for K_n/K is a piecewise linear function which satisfies the following properties:

1. *$\Phi_n(x)$ has $(V-1)n$ vertices,*

2. *the last (rightmost) vertex of* $\Phi_n(x)$ *has the same* x*-coordinate as the last vertex of* ϕ_n*,*

3. *the final slope (of the ray extending rightward from the last vertex) of* $\Phi_n(x)$ *is* $1/q^n$*,*

4. $\Phi_n(x)$ *coincides with* $\Phi_{n-1}(x)$ *for* x *smaller than the last coordinate of* Φ_{n-1}*,*

5. *the altitude of* $\Phi_n(x)$ *is strictly greater than the altitude of* Φ_{n-1} *and is unbounded as a function of* n*.*

Proof. By transitivity, $\Phi_n(x) = \Phi_{n-1} \circ \phi_n(x)$, so it is natural to proceed by induction. The base case is $\Phi_1 = \phi_1$, where there is nothing to prove: the shape of this function has been described explicitly already and satisfies all of the above conditions.

The first vertex of $\phi_n(x)$ is after the last vertex of $\Phi_{n-1}(x)$, and $\phi_n(x)$ is the identity up to its first vertex, so property (4) follows. The x-coordinate of the last vertex of $\phi_n(x)$ is after that of the last vertex of $\Phi_{n-1}(x)$, and so it remains the x-coordinate of the last vertex of $\Phi_n(x)$, verifying property (2). Moreover, after that point, we add $V-1$ new vertices, from those of ϕ_n, yielding (1). The final segment of ϕ_n corresponds to the final vertex $(q, 0)$ of $\mathcal{N}_n$, and hence has slope $\frac{1}{q}$. By inspection, the final slope of Φ_n is the product of the final slope of Φ_{n-1}, which is $\frac{1}{q^{n-1}}$, and the final slope of ϕ_n, which is $\frac{1}{q}$, so together the final slope is $\frac{1}{q^n}$, which is (3).

Finally, the altitude is the height of the last vertex of $\Phi_n(x)$, which lies over the last vertex of ϕ_n. By Proposition 5.10 combined with the limiting ramification data, we can express the x-coordinates of the last vertices of Φ_n and Φ_{n-1} as

$$Aq^n + B \quad \text{and} \quad Aq^{n-1} + B,$$

respectively, where A and B are positive constants which do not depend on n. The constant A comes from the part of the slope associated to the main terms, while B comes from the error terms plus the shift by $\operatorname{sgn}(v(\alpha_0))(d-1)v(\pi_n)$, and both incorporate the scaling by $e_{K/E}$.

Between these two vertices, the slopes of $\Phi_n(x)$ must be at least $\frac{p}{q^n}$ because the last (shallowest) slope is $\frac{1}{q^n}$ and the slopes are all powers of p. Then we can estimate the difference in altitudes as follows

$$\begin{aligned}\text{altitude}(\Phi_n) - \text{altitude}(\Phi_{n-1}) &\geq \frac{p}{q^n}\Big(Aq^n + B - (Aq^{n-1} + B)\Big) \\ &\geq Ap\left(1 - \frac{1}{q}\right).\end{aligned}$$

Thus the gap between consecutive altitudes is bounded below by a positive constant which does not depend on n, and so the altitudes are unbounded as n grows. □

With these preliminaries, our main theorem falls readily into place:

Theorem 5.18. *Suppose our branch, associated to the polynomial $f(x)$ and base point α_0, is strictly tamely ramification-stable over K. Then K_∞/K is arithmetically profinite, and there is a constant V such that for all n,*

$$K_n = K_\infty^{((V-1)n+1)}.$$

Proof. We first show that K_∞/K is arithmetically profinite. As explained in Wintenberger [41], we simply need a filtration of elementary extensions whose altitudes tend to infinity. Because Φ_n restricts to Φ_{n-1}, the elementary subextensions of K_n inside K_{n-1} are all of the elementary subextensions of K_{n-1}, which gives us our tower. The altitude of K_n tends to infinity by Proposition 5.17, hence the heights of these elementary subextensions do as well. From this we see that the extension is arithmetically profinite, and that its Hasse-Herbrand function $\Phi(x)$ is given by the pointwise limit of the intermediate Hasse-Herbrand functions $\Phi_n(x)$. Further, by Proposition 5.17, $\Phi_n(x)$ coincides with $\Phi_{n-1}(x)$ up to the last vertex of $\Phi_{n-1}(x)$, and so the same holds for $\Phi(x)$: whenever x is smaller than the x-coordinate of the last vertex of $\Phi_n(x)$, we have $\Phi(x) = \Phi_n(x)$.

The altitude of K_n over K is the same as the height of the $(V-1)n$th vertex of Φ, again by our assumption that the branch is strictly tamely ramification-stable. That altitude is strictly less than the height of the $((V-1)n+1)$th vertex of Φ, and so $K_n \subseteq K_\infty^{((V-1)n+1)}$. On the other hand, the final slope of $\Phi_n(x)$ is $\frac{1}{q^n}$, by Proposition 5.17. Since $\Phi(x) = \Phi_n(x)$ up to the the $((V-1)N+1)$th vertex, this is the same as the slope of $\Phi(x)$ going into the $((V-1)N+1)$th vertex, so the degree of $K_\infty^{((V-1)n+1)}$ over K is q^n, which is the same as the degree of K_n over K. Thus the two fields are equal, as claimed. □

Corollary 5.19. *Let $P(x)$ be a polynomial which either has degree p, or is post-critically bounded and has degree p^r. Take any nontrivial branch for $P(x)$, and suppose p does not divide the constant d associated to the branch. Then the dynamical branch extension K_∞/K is arithmetically profinite, and there are constants N and V such that after replacing K by K_N,*

$$K_n = K_\infty^{((V-1)(n-N)+1)},$$

for all n.

Proof. If $f(x)$ has prime degree or is post-critically bounded and $p \nmid d$, then any nontrivial branch associated to it is potentially strictly tamely ramification-stable by Proposition 5.15. Recall that this means that there is an N such that after restricting our branch to start at α_N it is strictly tamely ramification-stable over K_N.

To keep our indexing clear, set

$$\beta_n = \alpha_{N+n}, \quad L = K(\alpha_N), \quad L_n = L(\beta_n), \quad L_\infty = \bigcup L_n.$$

By definition, $L_\infty = K_\infty$. Then our main result, Theorem 5.18, applies to this branch, and so

$K_\infty = L_\infty$ as an extension of $L = K_N$ is arithmetically profinite and

$$L_n = L_\infty^{((V-1)n+1)}.$$

Translating from L to K, we see that $K_n = L_{n-N}$ if $n \geq N$. So making this change of index, we see that

$$K_n = L_{n-N} = L_\infty^{((V-1)(n-N)+1)}$$

for $n \geq N$, for the upper numbering relative to $L = K_N$. When $n < N$, $((V-1)(n-N)+1)$ is negative, which is handled by our convention for negative-indexed elementary subfields, that they are simply the ground field. Thus replacing K by K_N yields the claimed statement for all n. □

5.6 Applications and Effectivity

5.6.1 A question of Berger.

As our first application, we can offer a partial answer to a question raised by Berger [8]. That paper considers extensions of the same type studied here, though with two restrictions: the degree is the size of the residue field, and the base point is a uniformizer. An important intermediate result of that paper is the implication

$$K_\infty/K \text{ Galois} \quad \Rightarrow \quad K_\infty/K \text{ abelian}.$$

Berger asks if there is a more direct or elementary proof of this fact: the two proofs we are aware of, due to Berger [9] and Cais-Davis [11], use quite sophisticated machinery. Our results allow us to give such an elementary proof in some cases.

Let us outline Berger's use of this fact: if K_∞/K is abelian, then K_n/K is also abelian, and in particular normal. When K_n/K is normal and the degree of K_n/K_{n-1} is q, one can define, for each $\sigma \in \Gamma_K$, a power series $\mathrm{Col}_\sigma \in K[[T]]$ such that $\mathrm{Col}_\sigma(0) = 0$ and $\mathrm{Col}_\sigma(\alpha_n) = \alpha_n^\sigma$ (generalized Coleman power series). This power series commutes with f, and so by a result of Lubin [24], that power series is determined by the coefficient of its linear term, which gives a character from Γ_K to $\mathcal{O}_K^*$. This character is injective, because the action on the branch determines the action everywhere in the extension, since the branch generates the extension. Berger then goes on to study this character in detail.

But the logic flows the other way too: if we know that K_n/K is normal for some other reason, then we can construct these power series and the associated injective character, which would prove that K_∞/K is abelian. And indeed, the elementary subfields of K_∞ over K are all normal over K if K_∞/K is normal. Thus if one were to know that for all n there exists an m such that $K_n = K_\infty^{(m)}$ for some m, as in our main theorem, then K_∞/K must be abelian.

Theorem 5.20. *Assume p is odd. Suppose α_0 is a uniformizer for K, $f'(0)$ is nonzero, and we are given a branch associated to $f(x)$ and α_0 which is tamely ramification-stable.*

If K_∞/K is Galois, it is also abelian.

Proof. Because α_0 is a uniformizer, all of the polynomials $f^n(x) - \alpha_0$ are Eisenstein, so they are irreducible and give rise to a totally ramified extension of degree q^n. This means that $d = 1$ and that $[K_n : K_{n-1}] = q$ for all n.

The branch is strictly tamely ramification-stable, so we may apply Theorem 5.18, to conclude that for all n, the extension K_n/K is elementary, and therefore also Galois because K_∞/K is Galois.

Now let $\sigma \in \mathrm{Gal}(K_\infty/K)$. Because K_n/K is normal, α_n^σ is in $K_n = K(\alpha_n)$. The sequence $(\alpha_n^\sigma)_{n\in\mathbb{N}}$ is itself a branch, and by our assumption that p is odd and the irreducibility of $f(x) - \alpha_{n-1}$, we see that $N_{K_{n-1}}^{K_n}(\alpha_n) = \alpha_{n-1}$. This means that we can use Berger's construction ([8, Theorem 3.1]) to produce a uniquely determined series $\mathrm{Col}_\sigma \in \mathcal{O}_K[[T]]$ which acts by $\mathrm{Col}_\sigma(\alpha_n) = \alpha_n^\sigma$ and commutes with $f(x)$. This gives rise to a character χ from $\mathrm{Gal}(K_\infty/K)$ to $\mathcal{O}_K^*$ given by $\chi(\sigma) = \mathrm{Col}_\sigma'(0)$.

Since Col_σ commutes with $f(x)$ and $f'(0)$ is neither zero nor a root of unity, the series Col_σ is determined by $\mathrm{Col}_\sigma'(0)$ by Proposition 1.1 of Lubin [24]. Since Col_σ also determines the action of σ on α_n, and hence on the whole extension K_∞, the character χ is injective. Since $\mathrm{Gal}(K_\infty/K)$ embeds into an abelian group, it is itself abelian. □

Corollary 5.21. *Assume p is odd. Suppose α_0 is a uniformizer for K, $f'(0)$ is nonzero, and we are given a branch associated to $f(x)$ and α_0 which is potentially strictly tamely ramification-stable.*

If K_∞/K is Galois, it has a finite-index abelian subgroup.

Proof. Select N such that the branch is strictly tamely ramification-stable over K_N. Since it is still the case that the polynomials $f^n(x) - \alpha_0$ are Eisenstein, the new base point α_N remains a uniformizer. Therefore, Theorem 5.20 applies over this larger field, and hence $\mathrm{Gal}(K_\infty/K_N)$ is abelian. Its index in $\mathrm{Gal}(K_\infty/K)$ is exactly q^N. □

We cannot relax the assumption that α_0 is a uniformizer, as this is crucial to Berger's construction of the Coleman power series. Moreover, the fact that α_0 is a uniformizer means that every α_n will also be a uniformizer of the field it generates over K, and so $d = 1$ for any branch based at α_0. As a result, whether or not the branch is potentially strictly tamely ramification-stable depends *entirely* on $f(x)$.

Theorem 5.20 is not vacuous; there are strictly tamely ramification-stable branches associated to Galois extensions. For example, it is straightforward to check that Berger's example (Theorem 6.5 [8])

$$f(x) = x^3 + 6x^2 + 9x, \quad \alpha_0 = -3, \quad K = \mathbb{Q}_3$$

satisfies Theorem 5.20 by combining our observation that $d = 1$ with the effective results of Section 5.6.3.

In fact, because $d = 1$ and the polynomial in question has prime degree, the branch is guaranteed to be potentially strictly tamely ramification-stable, so we could have applied Corollary 5.21, without making any calculations, to determine that the Galois group has a large abelian subgroup (applying our effective results, one can see that this would have proven K_∞/K_1 is abelian). This can be done for many other examples involving a post-critically bounded or prime degree polynomial.

5.6.2 A question about wild ramification in arboreal extensions.

Both Aitken, Hajir, and Maire [1, Question 7.1] and Bridy, Ingram, Jones, Juul, Levy, Manes, Rubinstein-Salzedo, and Silverman [10, Conjecture 6] raise questions about wild ramification in arboreal extensions. Namely: are there arboreal extensions over number fields which are ramified at only finitely many primes but *not* wildly ramified?

We answer this negatively for all arboreal extensions associated to polynomials of prime-power degree. Under some restrictions on the base point, we can also show that such arboreal extensions are not only infinitely wildly ramified, but that all of their higher ramification subgroups are nontrivial. For the latter, we do not need the full strength of our results, only that K_∞/K is arithmetically profinite (which, for certain base points, already follows from Cais, Davis, and Lubin [12]).

Theorem 5.22. *Let F be a number field and $\mathfrak{p}$ a prime of F lying over a rational prime p. Let $P(x) \in \mathcal{O}_F[x]$ be a monic polynomial of degree p^r such that $f(x) \equiv x^{p^r} \mod \mathfrak{p}$, and let $\alpha_0 \in F$.*

Then the arboreal representation associated to $f(x)$ and α_0 is infinitely wildly ramified.

If, further, $f(x)$ has prime degree and $v(\alpha_0) \neq 0$, or is post-critically bounded with no restriction *on $v_{\mathfrak{p}}(\alpha_0)$, and there is a branch over α_0 whose associated constant d is not divisible by p, then every higher ramification subgroup over $\mathfrak{p}$ of the arboreal representation is nontrivial.*

Proof. It suffices to work over the completion K of F at a prime lying over p, and we may also take finite extensions of the ground field as necessary. Iteration and conjugation of PCB polynomials are PCB, so we may replace $f(x)$ by some conjugate iterate of itself, which allows us to modify its degree and ensure it fixes 0. So by Proposition 5.1 we may assume that in addition to being monic, $f(x)$ has integral coefficients, and fixes 0. Replacing $f(x)$ by $f^s(x)$ for a sufficiently large integer s, we may assume that the size of the residue field of K divides the degree of $f(x)$.

Recall that our results require $v(\alpha_0) \neq 0$. If $v(\alpha_0) = 0$, then after possibly extending F, we will conjugate by a translation to make its valuation positive. In particular, $f(x)$ has a fixed point congruent to α_0 modulo π_K, because

$$f(x) - x \equiv x^{p^r} - x \mod \pi_K,$$

and the size of the residue field divides p^r, so that every element of the residue field is a zero of $f(x) - x$ modulo π_K. Let α be such a fixed point, then replace $f(x)$ by its conjugate by $x \mapsto x - \alpha$ and α_0 by $\alpha_0 - \alpha$.

This leaves us with a final pair $(f(x), \alpha_0)$ where $v(\alpha_0) \neq 0$. It follows from Lemma 2.12 that (every) branch extension K_∞/K is infinitely wildly ramified, hence the full arboreal extension K_{arb}/K is also infinitely wildly ramified.

Because being post-critically bounded is conjugation and composition invariant, we may *always* assume when $f(x)$ is post-critically bounded that $v_{\mathfrak{p}}(\alpha_0) \neq 0$.

We can say more if $f(x)$ has prime degree with $v_{\mathfrak{p}}(\alpha_0) \neq 0$ or $f(x)$ is post-critically bounded and $v_{\mathfrak{p}}(\alpha_0) \neq 0$, and there is a branch such that $p \nmid d$, as then Corollary 5.19 applies: there is an N

such that after replacing K by K_N,

$$K_n = K_\infty^{((V-1)(n-N)+1)}.$$

Those are the subfields of K_∞ fixed by $\Gamma_K^{b_{(V-1)(n-N)+1}}$. The branch extension K_∞/K is contained in the full arboreal extension K_{arb}/K, which, combined with the functoriality of the upper numbering, means K_n is the subfield of K_∞/K that is fixed by the subgroup $\Gamma_{arb}^{b_{(V-1)(n-N)+1}}$. But the fields K_n are all distinct, and hence the subgroups which fix them must all be distinct too. Finally, it was shown that the ramification breaks $b_{(V-1)(n-N)+1}$ are unbounded as a function of n, and so every upper-numbered higher ramification subgroup of Γ_{arb} is nontrivial. □

Observation. Bridy, Ingram, Jones, Juul, Levy, Manes, Rubinstein-Salzedo, and Silverman [10] showed that a finitely ramified arboreal extension over a number field necessarily comes from a post-critically finite, and hence post-critically bounded map. This means that the preceding theorem applies as soon as one checks that p does not divide d (the stronger case, without restricting $v_{\mathfrak{p}}(\alpha_0)$ because the map is PCB).

The theorem tells us that, at least in some cases, the higher ramification subgroups of Γ_{arb} are all nontrivial, so we are led to wonder how large or small these subgroups might be. In particular, is K_{arb}/K arithmetically profinite? We suspect not, and conjecture that if there is no branch such that K_∞/K is Galois, then the wild ramification subgroup has infinite index inside Γ_{arb} (in other words, the tame part of K_∞/K has infinite degree over K). However, it seems plausible that this could be the only obstacle to the extension being arithmetically profinite: is it the case that for any $1 < \mu < \nu$, the index $[\Gamma_{arb}^\mu : \Gamma_{arb}^\nu]$ is finite?

5.6.3 Effective results; calculating limiting ramification data.

Every proof in this chapter, culminating in Corollary 5.19 is effective, and in practice straightforward to compute. Here we sketch the computation of most of the limiting ramification data (Definition 5.8). An implementation in SageMath [31] is available upon request. The only ineffective – but crucial – step made to obtain our results occured in Lemma 2.12. The determination of "sufficiently large" to ensure that (b) and (c) of this proposition are satisfied is not effective. This also means that the value $d = \lim_{n\to\infty} d_n$ is not effective. Knowing that p does not divide d is an important input to our main results, so from a computational perspective, this is a particularly unfortunate limitation.

However, if d is known, then all of our constants are effective. For example: if α_0 is a uniformizer, such as in the previous section, then $f^n(x) - \alpha_0$ is Eisenstein, so α_n is also a uniformizer, and so $d = 1$ and the pair is immediately tamely ramification-stable at the first level.

Calculating V, R, M, and E.

We begin with the computation of V, R, M, and E: the number of vertices, the (logarithm of) the x-coordinates of the vertices, and the main and error terms describing the heights of the vertices. Interestingly, these depend only on the valuations of the coefficients of $f(x)$ and on the *sign* of the valuation of α_0. They do not depend on the choice of branch.

All of the following steps can be extracted readily from the proof of Proposition 5.7. Roughly, the proposition tells us that when $v(\alpha_n)$ is small, we can drop the small error terms that show up in the minimum defining the Newton polygon $\mathcal{N}_n$ as long as we carefully track which terms achieve that minimum.

Step 1. For each $0 \leq k \leq r$, compute the minimum

$$M_{p^k} = \min_{p^k \leq j \leq q} \left\{ v\binom{j}{p^k} + v(f_j) \right\}. \tag{5.21}$$

Step 2. For each $0 \leq k \leq r$: if $v(\alpha_0)$ is positive (resp. negative), let j be the first (resp. last) index achieving the minimum (5.21) which defines M_{p^k}. Then set

$$E_{p^k} = j - p^k.$$

Step 3. Let $\mathcal{N}$ be the lower convex hull of the following vertices:

$$\left\{ \left(p^k, M_{p^k} + E_{p^k} \frac{1}{q^2} \right) \ : \ 0 \leq k \leq r \right\}.$$

The division by q^2 is arbitrary - any larger power of q will work as well. This polygon is an approximation to the polygons $\mathcal{N}_n$ that is precise enough to contain all the limiting ramification data. It is important that the error term is still present, because it can contribute vertices to the Newton polygons even though its contribution decreases rapidly. Degenerating all the way to the convex hull of the points (p^k, M_{p^k}) will lose this crucial information.

Step 4. Let V be the number of vertices of the polygon $\mathcal{N}$, and write the x-coordinates of the vertices of $\mathcal{N}$ as $p^{r_1}, ..., p^{r_V}$. Then the limiting ramification data is:

$$V(P, \alpha_0) = V$$

$$R(P, \alpha_0) = (r_1, ..., r_V)$$

$$M(P, \alpha_0) = (m_1, ..., m_V)$$

$$E(P, \alpha_0) = (e_1, ..., e_V)$$

(recall $m_i = M_{p^{r_i}}$, likewise $e_i = E_{p^{r_i}}$).

Calculating C.

The constant $C = \lim_{n\to\infty} q^n v(\alpha_n)$ is essentially valuation of α_n with respect to a valuation normalized for K_n. This constant requires slightly more information to calculate. Unlike V, R, M, and E, this constant depends on the branch. However, the dependence is weaker than one might expect.

Proposition 5.23. *If $\alpha_0 \neq 0$ and the valuation of the base point $v(\alpha_0)$ is fixed, then there is a constant N which is uniform among all branches such that $C = q^N v(\alpha_N)$. In fact, the constant N depends only on the degree of f.*

Assuming $\alpha_0 \neq 0$, then we can give an explicit upper bound for the constant N: if $v(\alpha_0) = 0$, then we may take $N = 0$, and if $v(\alpha_0) > 0$ then we may take $N = v(\alpha_0)$.

If $\alpha_0 = 0$, but we truncate all branches to remove their leading zeros, there is such a constant N, which now depends on the valuations of the coefficients of f. In this case, let k be the number of leading zeros in the branch and let $\ell = \max\{v(f_j)\}$. Then we may take $N = k + \ell$.

Proof. The claims follow from a closer inspection of the proof of Lemma 2.12. The lemma guarantees that there is an N such that the constant C is given by $q^N v(\alpha_N)$; a priori this N depends on the branch, but we will show it does not. If $v(\alpha_0) < 0$ then we are immediately done with $N = 0$.

If $\alpha_0 \neq 0$, then the decrease in valuation is partly controlled by the following estimate:

$$v(\alpha_n) \leq \max\{v(\alpha_{n-1}) - 1, v(\alpha_{n-1})/2\}.$$

In the maximum, it is easy to see that

$$v(\alpha_{n-1}) - 1 \leq v(\alpha_{n-1})/2$$

if and only if

$$v(\alpha_{n-1}) \leq 2,$$

and when that occurs, it must be that $v(\alpha_n) \leq 1$. So after $N = v(\alpha_0)$ steps, we are guaranteed to be in a situation where Lemma 2.12(a) applies, and hence $C = q^N v(\alpha_N)$.

Otherwise, $\alpha_0 = 0$. Let k be the number of leading 0s in the branch, which means $\alpha_k \neq 0$ and $\alpha_{k-1} = 0$, and by inspecting the Newton polygon of $f(x) - \alpha_{k-1} = f(x)$, a generous bound for $v(\alpha_k)$ is $\ell = \max\{v(P_j)\}$, as long as $\alpha_1 \neq 0$. Then we may apply our reasoning for the case $\alpha_0 \neq 0$, but with α_k in place of α_0 to see that

$$C = q^{k+\ell} v(\alpha_{k+\ell}).$$

□

With 5.23, we can explicitly determine a value N depending (mildly) on the branch and f such that $C = q^N v(\alpha_n)$.

Sample calculation.

In any particular case, it is mostly straightforward to check that a pair is tamely ramification-stable, *except* for the tameness component, since we do not have an effective way to compute d. However, it is possible to do so in some cases.

The following example is small enough that we can carry out the calculation by hand.

Let $K = E = \mathbb{Q}_3(\sqrt{3})$ with valuation v normalized so that $v(\sqrt{3}) = 1$. Consider the polynomial

$$f(x) = x^9 + 12\sqrt{3}x^7 + 18x^6 + 3\sqrt{3}x^4 + \frac{3}{5}x^3 + 9x,$$

with any branch whose initial sequence of valuations looks like $(4, 2/3, 2/27, ...)$.

We readily obtain our effective constants:

$$V = 3,$$

$$R = (0, 1, 2),$$

$$M = (3, 2, 0),$$

$$E = (3, 0, 0).$$

as well as

$$C = 9^4 v(\alpha_4) = 9^4 \cdot \frac{2}{3} \cdot \frac{1}{9^3} = 6.$$

Inspecting the first few levels of such a branch in Sage, we see that our sequence d_n looks like $4, 2, 2, ...$, hence $d = 2$, which is not divisible by $p = 3$. To be more precise, while the value N from Lemma 2.12 is not effectively determined, we can see from the proof that as soon as some d_n is not divisible by p and $v(\alpha_n) < 1$, we have reached a suitable index. This is because at each step there is no way for the valuation to decrease by a factor of q without the ramification index being q as well. Combined with this limiting ramification data, one can see directly that (f, α_1) is tamely ramification-stable. Therefore, when we replace K by K_1, we may apply Theorem 5.18 to obtain

$$K_n = K_\infty^{((V-1)(n-1)+1)}.$$

So, even though $f(x)$ is not prime degree or post-critically bounded, it is an example of a polynomial whose branch extensions are amenable to study by our methods.

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